TRIUMF DBD workshop

Interfacing theory and experiment for reliable DBD NMEs calculation

Vancouver, Canada, May 11-13, 2016

$2\nu\beta\beta$-decay is the key for reliable calculation of $0\nu\beta\beta$-decay NMEs

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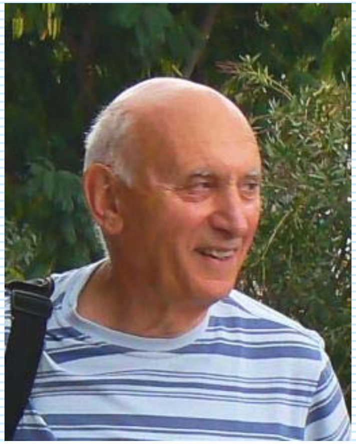
OUTLINE

I. Some notes about $2\nu\beta\beta$-decay
II. The DBD Nuclear Matrix Elements and the SU(4) symmetry
III. QRPA for description of states of multiphonon origin
IV. How many $0\nu\beta\beta$-decay NMEs we need to calculate?



November 1984, Dubna

We need reliable calculation of DBD NMEs

Some notes about $2\nu\beta\beta$-decay (NMEs)

Both $2\nu\beta\beta$ and $0\nu\beta\beta$ operators connect the same states.
Both change two neutrons into two protons.

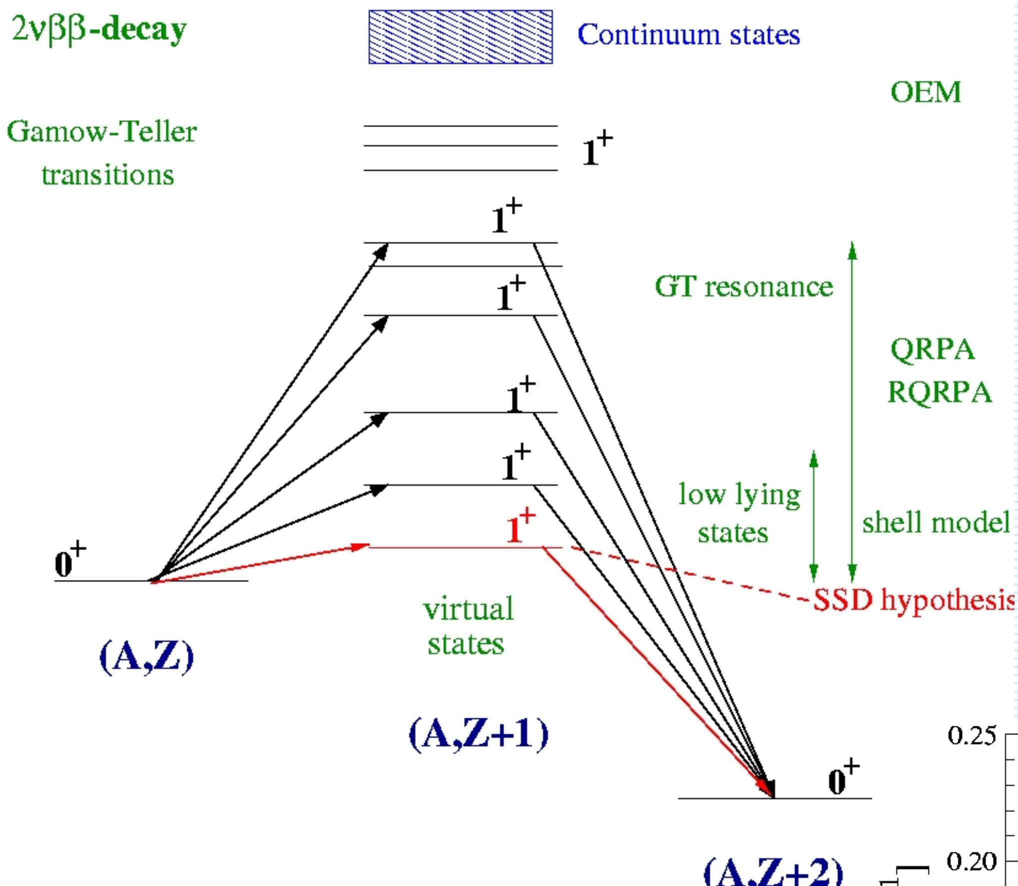
Explaining $2\nu\beta\beta$-decay is necessary but not sufficient

There is no reliable calculation of the *$2\nu\beta\beta$-decay NMEs*

Calculation via intermediate nuclear states: QRPA (sensitivity to pp-int.)
ISM (quenching, truncation of model space, spin-orbit partners)

Calculation via closure NME: IBM, PHFB

No calculation: EDF



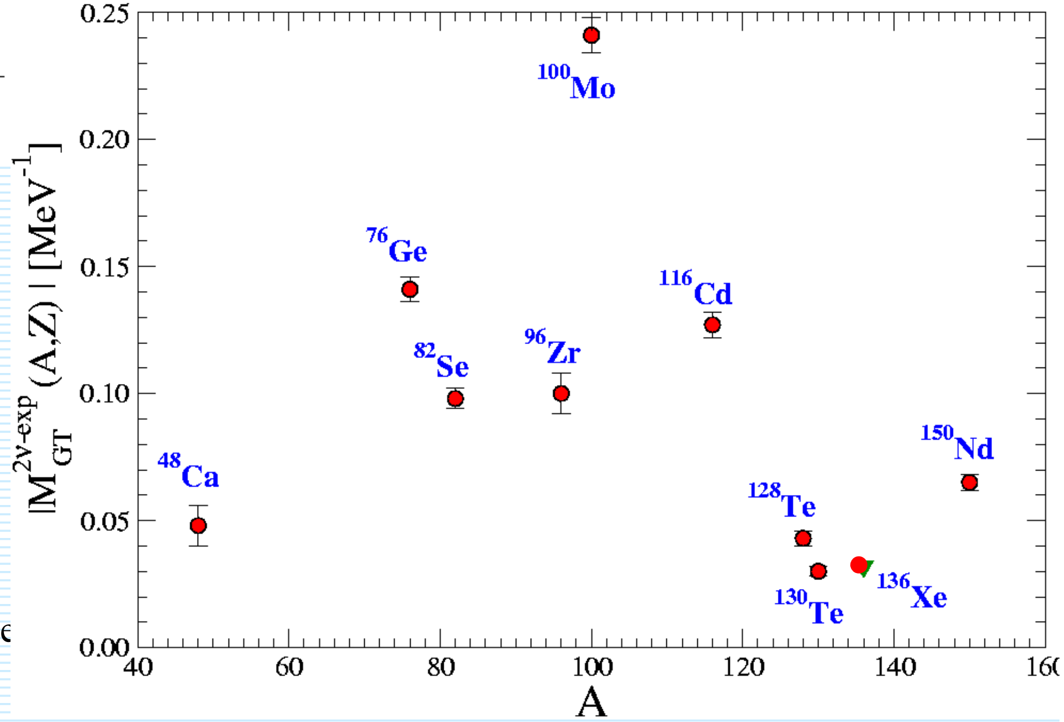
2νββ-decay nuclear matrix elements

$$\frac{1}{T_{1/2}^{2\nu-exp}} = G^{2\nu}(E_0, Z)\; g_A^4\; |M_{GT}^{2\nu}|^2$$

Deduced from measured $T_{1/2}^{2\nu}$

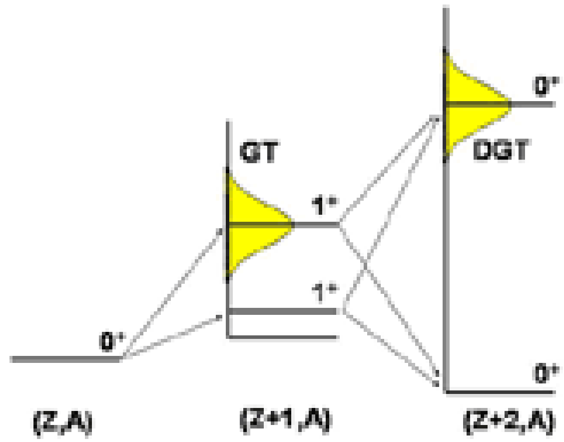
$$M_{GT}^{2\nu} = \sum_m \frac{< 0_f^+ ||\tau^+\sigma|| 1_m^+ >< 1_m^+ ||\tau^+\sigma|| 0_i^+ >}{E_m - E_i + \Delta}$$

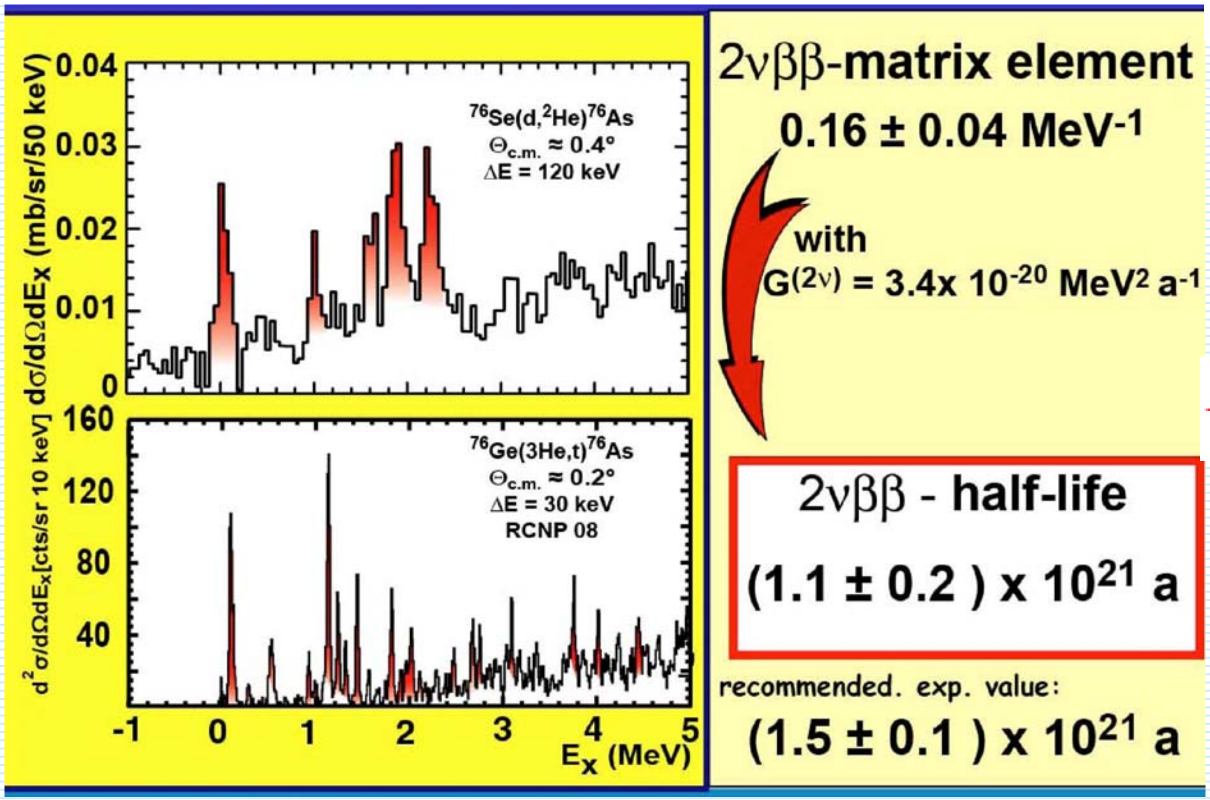
Differencies in NME: by factor ~ 10



The cross sections of (t,^{3}He) and (d,^{2}He) reactions give B(GT$^\pm$) for β^+ and β^-, product of the amplitudes (B(GT)$^{1/2}$) entering the numerator of $M^{2\nu}_{GT}$

$$M^{2\nu}_{GT} = \sum_m \frac{M^{(+)}_{GT}(m)\ M^{(-)}_{GT}(m)}{Q_{\beta\beta}/2 + m_e + E_x(1^+_m) - E_0}$$





Grewe, …Frekers at al, PRC 78, 044301 (2008)

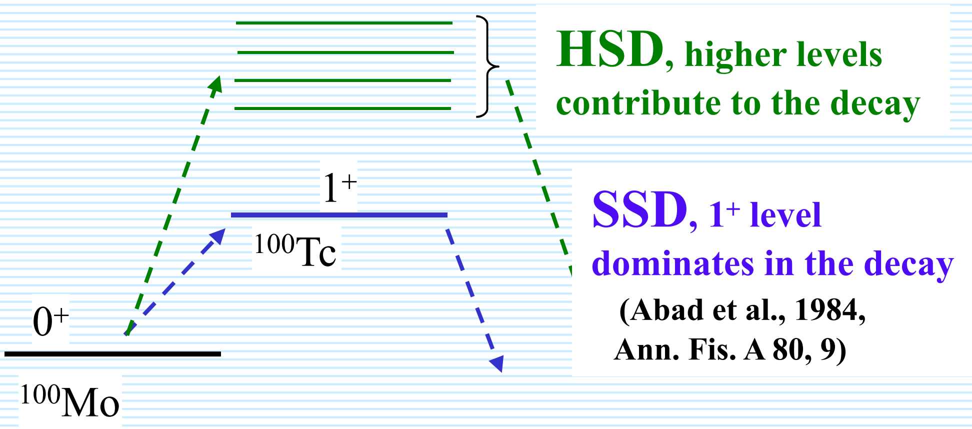
Closure 2 νββ-decay NME

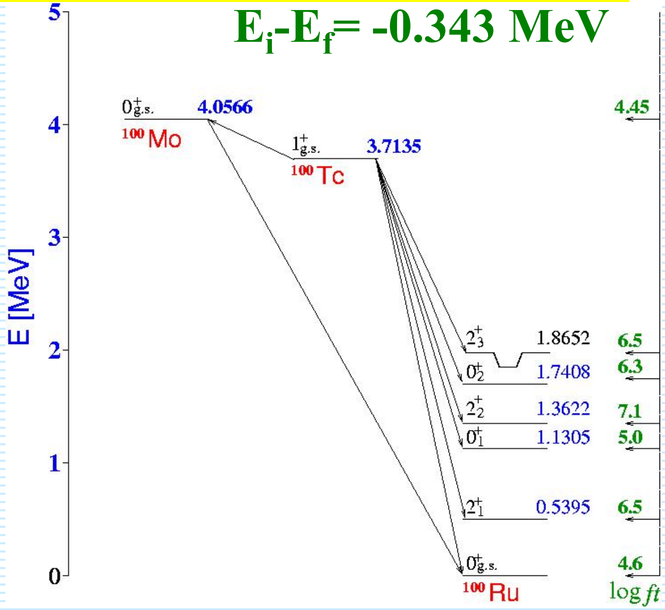
$$M^{2\nu}_{GT-cl} = \sum_m M^{(+)}_{GT}(m)\ M^{(-)}_{GT}(m)$$

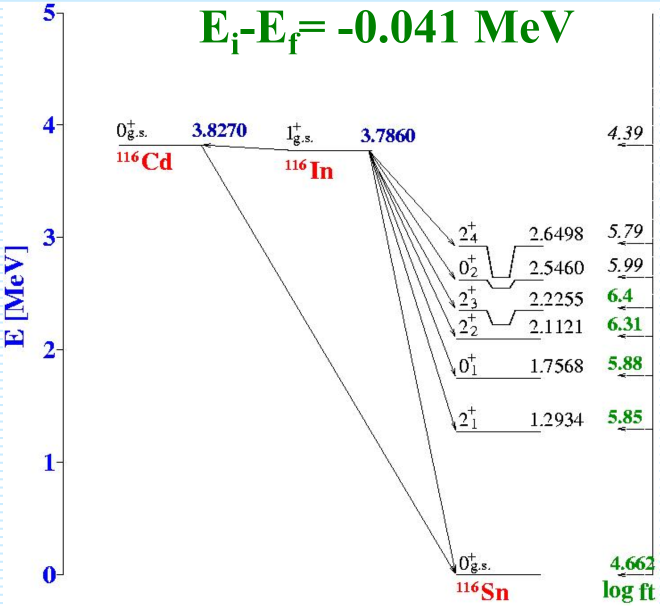
SSD hypothesis

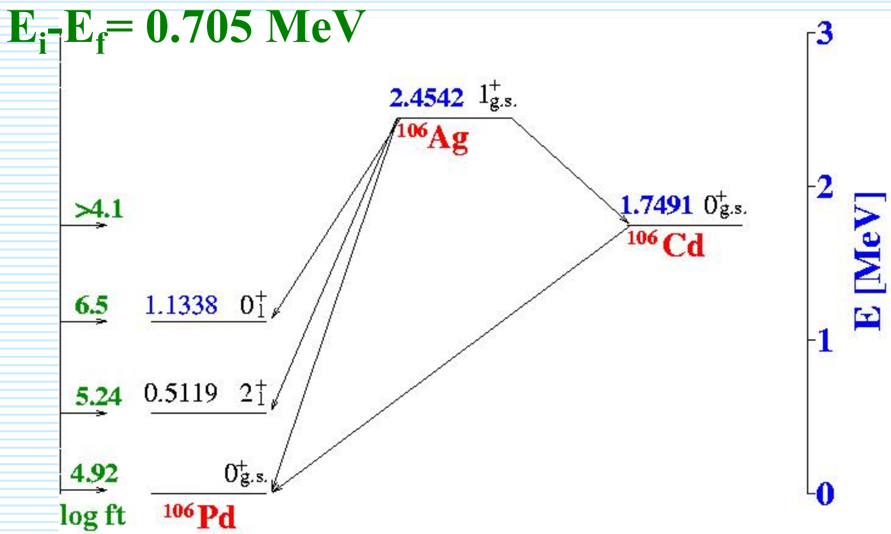
$$g_A^2\ M^{2\nu}_{GT-cl} = \frac{3\ D}{\sqrt{ft_{EC}\, ft_{\beta^-}}}$$

Single State Dominance (^{100}Mo, ^{106}Cd, ^{116}Cd, ^{128}Te …)









SSD – theoretical studies

$$M_{GT}^{K} = \sum_m \left(\frac{M_m^i(1^+) M_m^f(1^+)}{E_m - E_i + e_{10} + \nu_{10}} + \frac{M_m^i(1^+) M_m^f(1^+)}{E_m - E_i + e_{20} + \nu_{20}} \right) \qquad M_{GT}^{K} = M_{GT}^{L}(\nu_{10} \leftrightarrow \nu_{20})$$

SSD

$$\Rightarrow \frac{M_1^i(1^+) M_1^f(1^+)}{E_1 - E_i + e_{10} + \nu_{10}} + \frac{M_1^i(1^+) M_1^f(1^+)}{E_1 - E_i + e_{20} + \nu_{20}} \Rightarrow 2\frac{M_1^i(1^+) M_1^f(1^+)}{E_1 - E_i + \Delta}$$

common approx.

$$e_{10} + \nu_{10} \approx e_{20} + \nu_{20} \approx (E_i - E_f)/2 \equiv \Delta$$

Isotope	f.s.	$T_{1/2}$(SSD)[y]	$T_{1/2}$(exp.)[y]
		$2\nu\beta^-\beta^-$	
^{100}Mo	$0_{g.s.}$	$6.8\ 10^{18}$	$6.8\ 10^{18}$
	0_1	$4.2\ 10^{20}$	$6.1\ 10^{18}$
^{116}Cd	$0_{g.s.}$	$1.1\ 10^{19}$	$2.6\ 10^{19}$
^{128}Te	$0_{g.s.}$	$1.1\ 10^{25}$	$2.2\ 10^{24}$
		EC/EC	
^{106}Cd	$0_{g.s.}$	$>4.4\ 10^{21}$	$>5.8\ 10^{17}$
^{130}Ba	$0_{g.s.}$	$5.0\ 10^{22}$	$4.0\ 10^{21}$

E_1-E_i ≈ 0 or neg. $\Rightarrow$ sensitivity to lepton energies in energy denominators

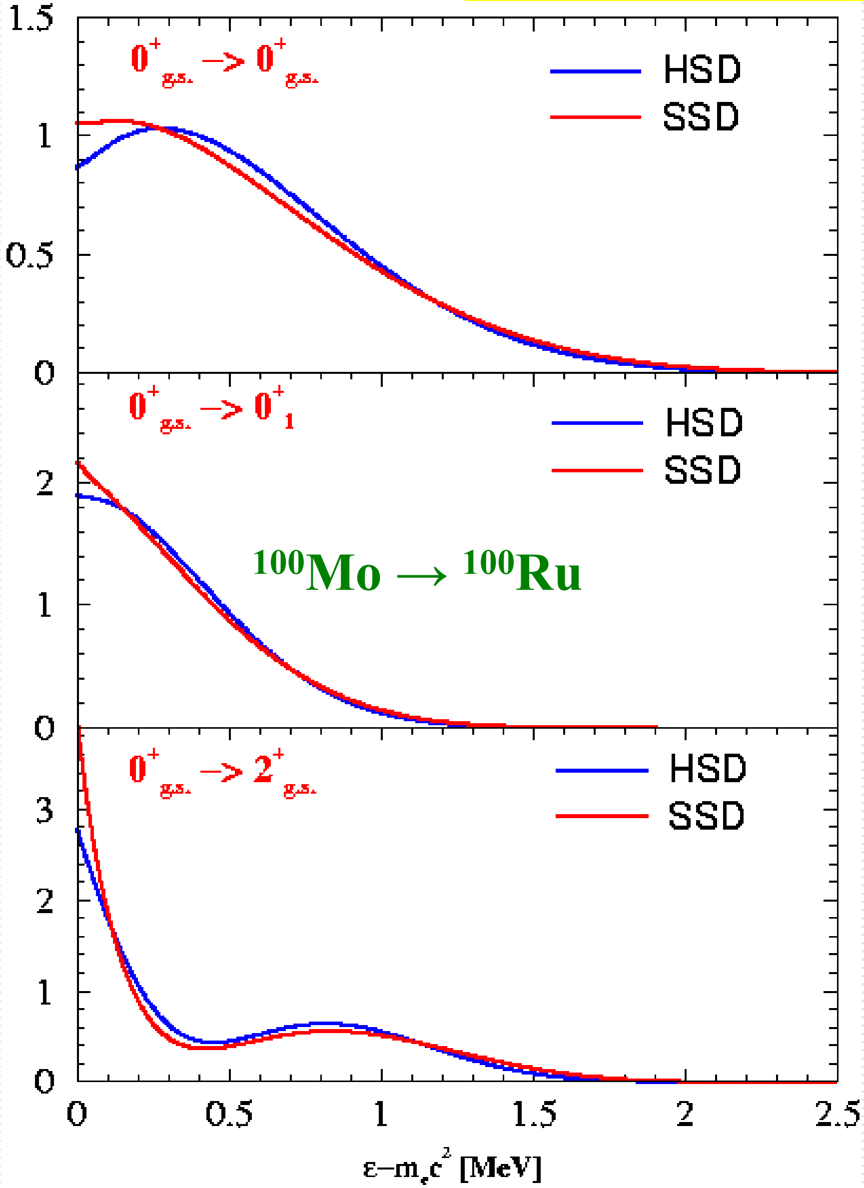
$\Rightarrow$ SSD and HSD offer different differential characteristics

Šimkovic, Šmotlák, Semenov
J. Phys. G, 27, 2233, 2001

Domin, Kovalenko, Šimkovic, Semenov, NPA 753, 337 (2005)

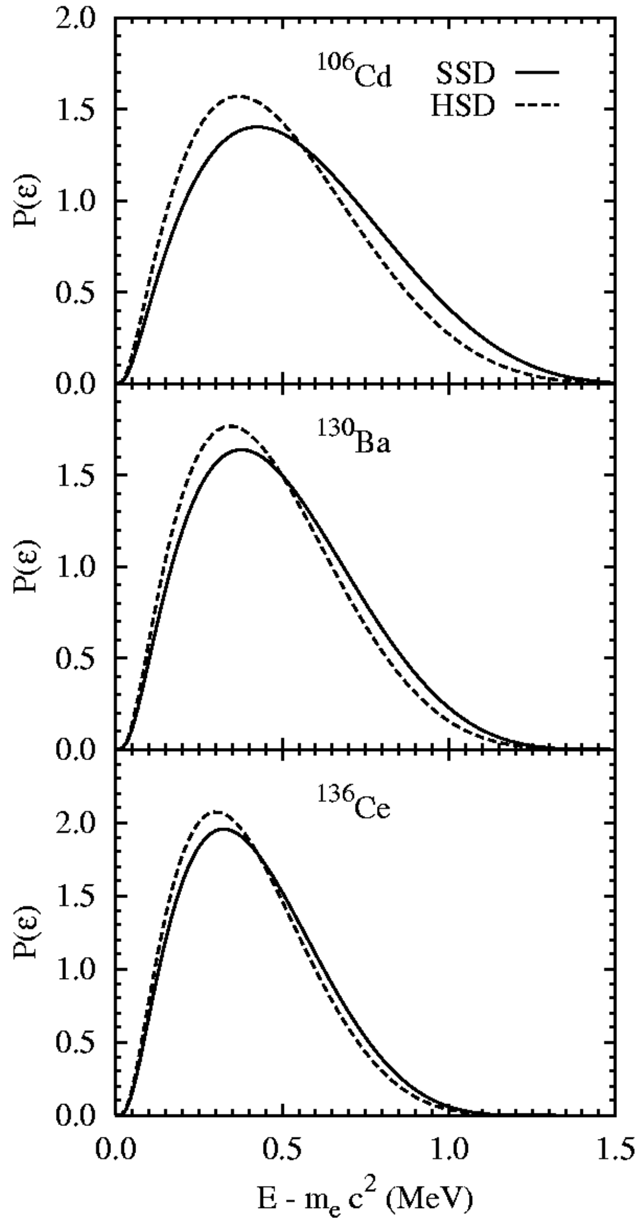
SSD differential characteristics

$2\nu\beta^-\beta^-$–decay



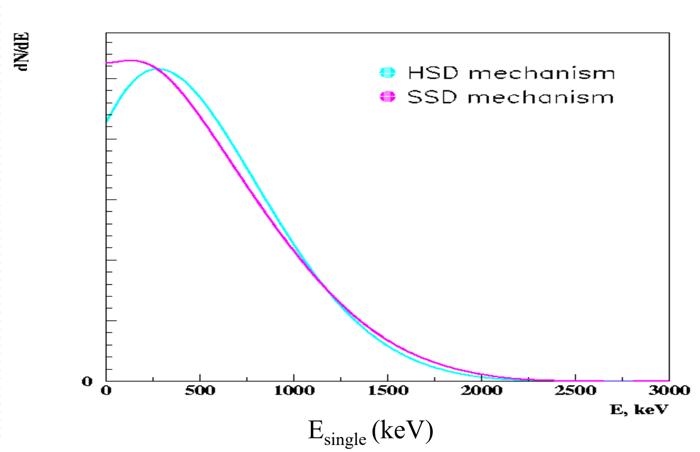
Do not depend on M^iM^f

$2\nu EC/\beta^+$–decay



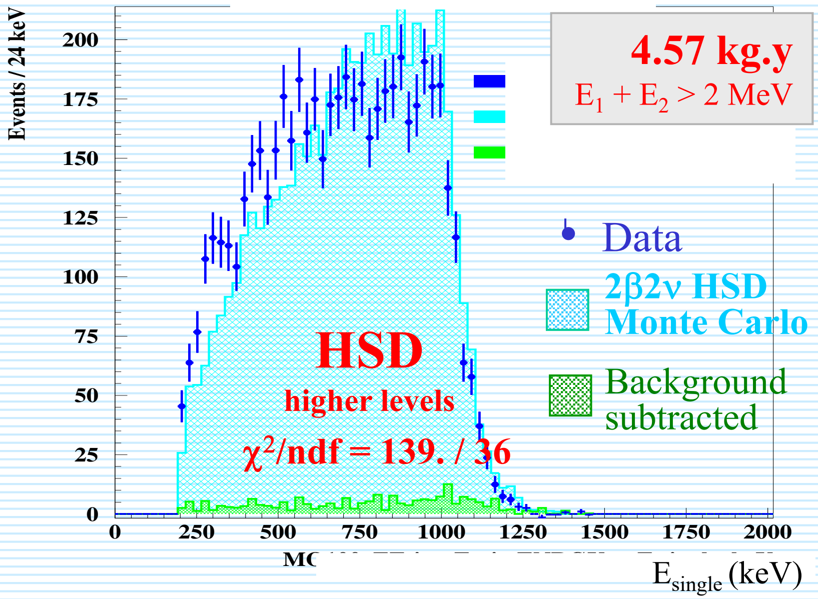
^{100}Mo 2β2ν: Experimental Study of SSD Hypothesis

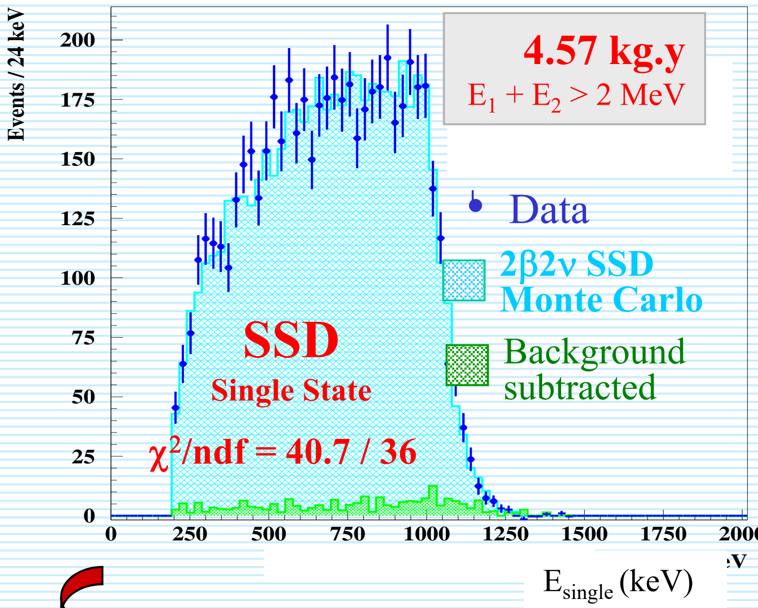
NEMO 3 exp.



Single electron spectrum different between SSD and HSD

Šimkovic, Šmotlák, Semenov J. Phys. G, 27, 2233, 2001





HSD: $T_{1/2} = 8.61 \pm 0.02$ (stat) ± 0.60 (syst) $\times 10^{18}$ y

SSD: $T_{1/2} = 7.72 \pm 0.02$ (stat) ± 0.54 (syst) $\times 10^{18}$ y

^{100}Mo 2β2ν single energy distribution in favour of Single State Dominant (SSD) decay

2νββ-decay rate

$$\left[T_{1/2}^{2\nu\beta\beta}(0^+)\right]^{-1} = \frac{m_e}{8\pi^7 \ln 2}(G_\beta m_e^2)^4 I^{2\nu}(0^+),$$

$$I^{2\nu}(0^+) = \frac{1}{m_e^9}\int_{m_e}^{E_i-E_f-m_e} F_0(Z_f, E_{e_1}) p_{e_1} E_{e_1} dE_{e_1} \times \int_{m_e}^{E_i-E_f-E_{e_1}} F_0(Z_f, E_{e_2}) p_{e_2} E_{e_2} dE_{e_2} \times \int_0^{E_i-E_f-E_{e_1}-E_{e_2}} E_{\nu_1}^2 E_{\nu_2}^2 \mathcal{A}^{2\nu} dE_{\nu_1}.$$

$$\mathcal{A}^{2\nu} = g_V^4\left[\frac{1}{4}|M_F^K + M_F^L|^2 + \frac{3}{4}|M_F^K - M_F^L|^2\right] - g_V^2 g_A^2 \mathrm{Re}\left\{M_F^{K*} M_{GT}^L + M_{GT}^{K*} M_F^L\right\} + \frac{g_A^4}{3}\left[\frac{3}{4}|M_{GT}^K + M_{GT}^L|^2 + \frac{1}{4}|M_{GT}^K - M_{GT}^L|^2\right]$$

$$M_F^K = \sum_n \frac{K(0_n^+)}{2} F_n, \quad M_F^L = \sum_n \frac{L(0_n^+)}{2} F_n,$$

$$M_{GT}^K = \sum_n \frac{K(1_n^+)}{2} G_n, \quad M_{GT}^L = \sum_n \frac{L(1_n^+)}{2} G_n,$$

$$F_n = \langle 0_f^+ \| \sum_m \tau_m^- \| 0_n^+\rangle\langle 0_n^+ \| \sum_m \tau_m^- \| 0_i^+\rangle,$$

$$G_n = \langle 0_f^+ \| \sum_m \tau_m^- \sigma_m \| 1_n^+\rangle\langle 1_n^+ \| \sum_m \tau_m^- \sigma_m \| 0_i^+\rangle$$

$$K_n(J^+) = \frac{2}{(2E_n(J^+) - E_i - E_f) + \epsilon_K} + \frac{2}{(2E_n(J^+) - E_i - E_f) - \epsilon_K}$$

$$L_n(J^+) = \frac{2}{(2E_n(J^+) - E_i - E_f) + \epsilon_L} + \frac{2}{(2E_n(J^+) - E_i - E_f) - \epsilon_L}$$

$$\epsilon_K = E_{e_2} + E_{\nu_2} - E_{e_1} - E_{\nu_1}$$

$$\epsilon_L = E_{e_1} + E_{\nu_2} - E_{e_2} - E_{\nu_1}$$

In the limit $2E_n - E_i - E_f = 0$ ⟹ $\mathcal{A}^{2\nu} = 0$

2νββ-decay within the field theory

F.Š., G. Pantis, Phys. Atom. Nucl. 62 (1999) 585

Weak interaction Hamiltonian

$$\mathcal{H}^{\beta}(x) = \frac{G_F}{\sqrt{2}} 2 \left[\bar{e}_L(x)\gamma_\alpha \nu_{eL}(x)\right] j_\alpha(x) + h.c.$$

2nbb-decay amplitude

$$< f|S^{(2)}|i > =$$
$$\frac{(-i)^2}{2}\left(\frac{G_F}{\sqrt{2}}\right)^2 L_{\mu\nu}(p_1,p_2,k_1,k_2) J_{\mu\nu}(p_1,p_2,k_1,k_2)$$
$$-(p_1 \leftrightarrow p_2) - (k_1 \leftrightarrow k_2) + (p_1 \leftrightarrow p_2)(k_1 \leftrightarrow k_2)$$

Hadron part of amplitude

$$J_{\mu\nu}(p_1,p_2,k_1,k_2) = \int e^{-i(p_1+k_1)x_1} e^{-i(p_2+k_2)x_2}$$
$$_{out}<p_f|T(J_\mu(x_1)J_\nu(x_2))|p_i>_{in}\, dx_1 dx_2$$

Integral representation of M_{GT}

$$M_{GT} = \frac{i}{2}\int_0^\infty (e^{i(p_{10}+k_{10}-\Delta)t} + e^{i(p_{20}+k_{20}-\Delta)t}) M_{AA}(t)dt$$

with

$$M_{AA}(t) =< 0_f^+|\frac{1}{2}[A_k(t/2), A_k(-t/2)]|0_i^+ >$$

$$A_k(t) = e^{iHt}A_k(0)e^{-iHt}, \qquad A_k = \sum_i \tau_i^+(\vec{\sigma}_i)_k, \;\; k = 1, 2, 3.$$

$$A_k(t) = e^{itH}A_k(0)e^{-itH} = \sum_{n=0}^\infty \frac{(it)^n}{n!}\overbrace{[H[H...[H}^{n\ times}, A_k(0)]...]]$$

Completeness:

Σ_n |n><n|=1

$$< A'|J_\alpha(x_1)J_\beta(x_2)|A > \;\; = \;\; \sum_n < A'|J_\alpha(0,\vec{x}_1)|n >< n|J_\beta(0,\vec{x}_2)|A > \times$$
$$e^{-i(E'-E_n)x_{10}}e^{-i(E_n-E)x_{20}}$$

$$\int_0^\infty e^{-iat}dt \Rightarrow \lim_{\epsilon\to 0}\int_0^\infty \; e^{-i(a-i\epsilon)t}dt = \lim_{\epsilon\to 0}\frac{-i}{a-i\epsilon}$$

$$M_{GT} = \sum_n \frac{< 0_f^+|A(0)_k|1_n^+ >< 1_n^+|A(0)_k|0_i^+ >}{E_n - E_i + \Delta}$$

Double beta decay is a two-body process

$$\mathbf{H = one - body + two - body}, \quad \mathrm{A}_k(0) = \text{one–body}$$

$$A_k(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \overbrace{[H[H...[H}^{n\ times}, A_k(0)]...]]$$

$$\begin{array}{lllllllll}
\text{n=0} & [H^{(0)} A] & [1] & & & & & & \\
 & & \downarrow & \searrow & & & & & \\
\text{n=1} & [H^{(1)}, A] & [1] & & [2] & & & & \\
 & & \downarrow & \searrow & \downarrow & \searrow & & & \\
\text{n=2} & [H^{(2)}, A] & [1] & & [2] & & [3] & & \\
 & & \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
\text{n=3} & [H^{(3)}, A] & [1] & & [2] & & [3] & & [4]
\end{array}$$

$$If \quad \mathrm{H} \approx \text{one–body op.} \Longrightarrow \mathbf{A_k(t)} \text{ is one–body op.}$$

Operator Expansion Method and DBD NMEs

C.R. Ching, T.H. Ho, Commun. Theor. Phys. 10, 45 (1988); 11, 433 (1989); 11, 495 (1989)

$$M^{2\nu}_{GT} = \frac{1}{2\Delta}\sum_n \langle f|\mathcal{O}_{GT}|n\rangle \, (1- \frac{(E_n - E_i) - (E_f - E_n)}{\Delta} + \frac{(E_n - E_i)^2 - (E_f - E_n)^2}{\Delta^2} - \cdots) \, \langle n|\mathcal{O}_{GT}|i\rangle$$

$$= \frac{1}{2\Delta}\left(\langle f|[\mathcal{O}_{GT},\mathcal{O}_{GT}]|i\rangle + \frac{\langle f|[\mathcal{O}_{GT},[H,\mathcal{O}_{GT}]]|i\rangle}{\Delta} + \frac{\langle f|[\mathcal{O}_{GT},[H,[H,\mathcal{O}_{GT}]]]|i\rangle}{\Delta^2} + \cdots\right).$$

Convergence of a series?

F. Š., JINR Commun. 39, 21 (1989); M. Gmitro, F. Š., Izv. AN SSR 54, 1780 (1990);
F. Š., G. Pantis, Czech. J. Phys. B 48, 235 (1998); A. Faessler, F. Š., J. Phys. G 24, 2139 (1998)

$$M^{2\nu}_{GT} = \langle f|\frac{i}{2}\int_0^\infty e^{i\Delta t}[\mathcal{O}_{GT}(0),\mathcal{O}_{GT}(t)]dt|i\rangle$$

$$\mathcal{O}_{GT}(t) = e^{iHt}\mathcal{O}_{GT}(0)e^{-iHt}$$

This problem does not appear?

Nuclear Hamiltonian

$$H \approx V_C + V_{CN} + V_{TN}$$

Central and tensor nuclear interactions

$$V_{CN} = \frac{1}{2}\sum_{i\neq j} [\ (g_{SE}(r_{ij})\ \Pi^r_e(ij) \ + \ g_{SO}(r_{ij})\ \Pi^r_o(ij) \ )\ \Pi^\sigma_s(ij) + (g_{TE}(r_{ij})\ \Pi^r_e(ij) \ + \ g_{TO}(r_{ij})\ \Pi^r_o(ij) \ )\ \Pi^\sigma_t(ij) \ ],$$

Effective Coulomb int. due to different ground states

$$V_C = \frac{1}{2}\sum_{i\neq j} (E_f - E_i)\ O^\tau_{ij}$$

$$V_{TN} = \frac{1}{2}\sum_{i\neq i} (\ g_{TNE}(r_{ij})\ \Pi^r_e(ij) + g_{TNO}(r_{ij})\ \Pi^r_o(ij)\ )\ S_{ij}.$$

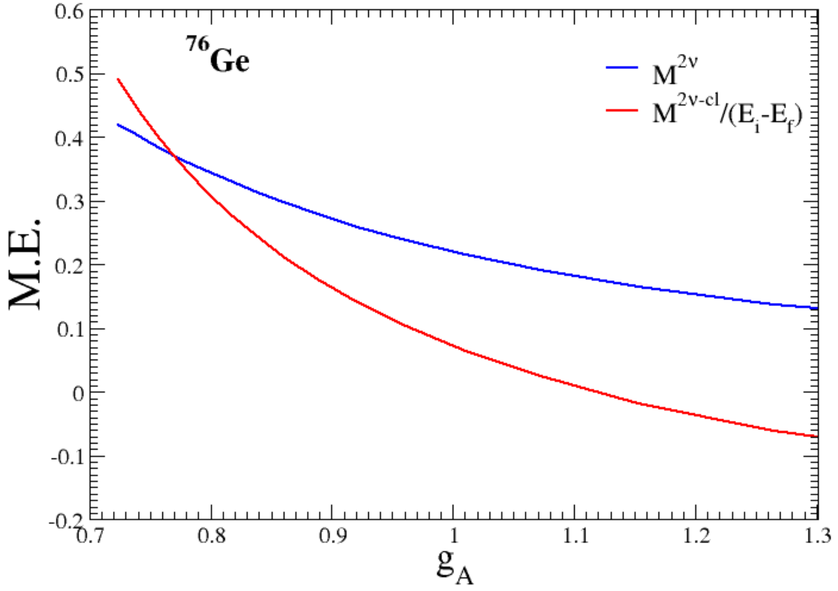
2νββ NME within the OEM

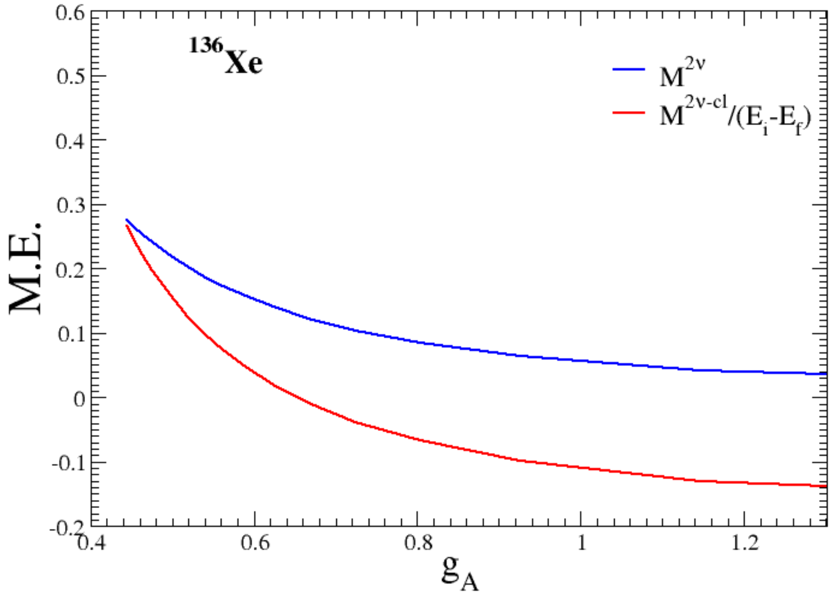
$$M_{GT} = <0^+_f|\frac{1}{2}\mathcal{P}\sum_{i\neq j}\tau^+_i\tau^+_j\ (\ \mathcal{V}^{singlet}(r_{ij})\ \Pi^\sigma_s(ij)\ + \mathcal{V}^{triplet}(r_{ij})\ \Pi^\sigma_t(ij)\ +\ \mathcal{V}^{tensor}(r_{ij})\ S_{ij}\ )\ |0^+_i>$$

$$\mathcal{V}^{singlet} = \frac{-2}{g_{TE} - g_{SE} - 4g_{TNE} + \Delta} - \frac{4}{g_{TE} - g_{SE} + 2g_{TNE} + \Delta}$$

$$\mathcal{V}^{triplet} = \frac{1}{3}[\frac{4}{\Delta} + \frac{4}{-6g_{TNO} + \Delta} + \frac{4}{6g_{TNO} + \Delta} - \frac{2}{g_{SO} - g_{TO} + 4g_{TNO} + \Delta} - \frac{4}{g_{SO} - g_{TO} - 2g_{TNO} + \Delta}]$$

$$\mathcal{V}^{tensor} = \frac{1}{3}[\frac{1}{\Delta} + \frac{1}{-6g_{TNO} + \Delta} - \frac{2}{6g_{TNO} + \Delta} + \frac{1}{g_{SO} - g_{TO} + 4g_{TNO} + \Delta} - \frac{1}{g_{SO} - g_{TO} - 2g_{TNO} + \Delta}]$$





If central and tensor interactions are neglected we end up with closure NME with

$\langle E_n-(E_i+E_f)/2\rangle = E_i - E_f = \Delta$

$$M_{GT} = \langle 0_f^+ | \frac{1}{2} \sum_{n \neq m} \tau_n^+ \tau_m^+ \frac{2}{\Delta} \vec{\sigma}_n \cdot \vec{\sigma}_m | 0_i^+ \rangle$$

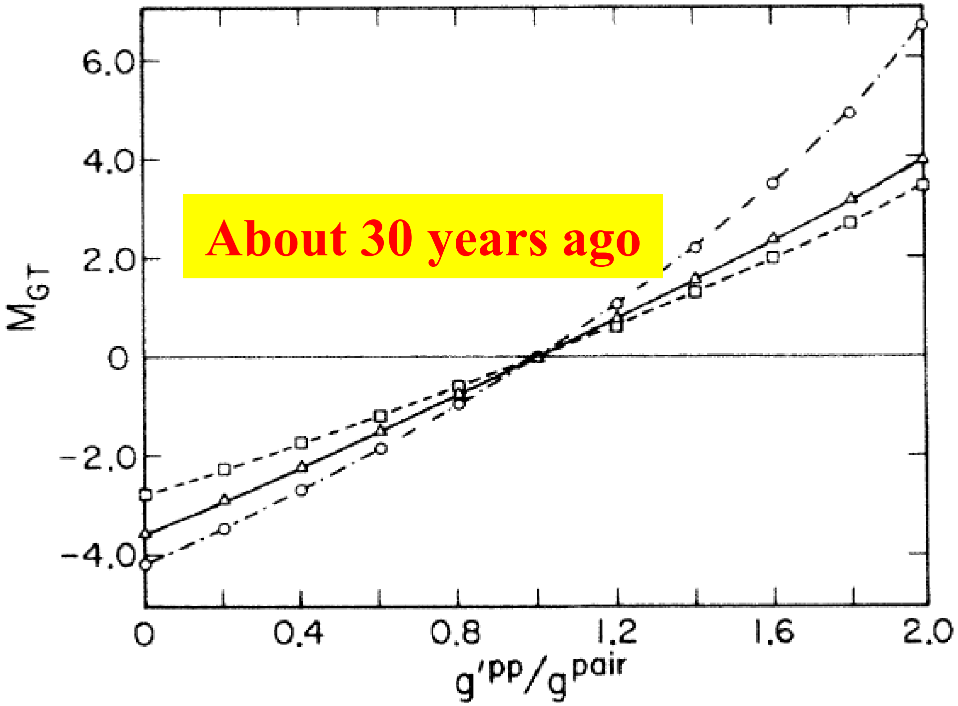
The DBD Nuclear Matrix Elements and the SU(4) symmetry

Suppression of the DBD NMEs and their sensitivity to particle particle interaction strength

Suppression of the Two Neutrino Double Beta Decay by Nuclear Structure Effects
P. Vogel, M.R. Zirnbauer, PRL (1986) 3148

O. Civitarese, A. Faessler, T. Tomoda, PLB 194 (1987) 11
E. Bender, K. Muto, H.V. Klapdor, PLB 208 (1988) 53

…



The isospin is known to be a good approximation in nuclei

In heavy nuclei the SU(4) symmetry is strongly broken by the spin-orbit splitting.

What is beyond this behavior? Is it an artifact of the QRPA?

s.p. mean-field

Conserves SU(4) symmetry

$$H = \underbrace{e_n N_n + e_p N_p - g_{pair}\left(\sum_{M_T=-1,0,1} A^\dagger_{0,1}(0,M_T)A_{0,1}(0,M_T) + \sum_{M_S=-1,0,1} A^\dagger_{1,0}(M_S,0)A_{1,0}(M_S,0)\right) + g_{ph}\sum_{a,b} E^\dagger_{a,b}E_{a,b}}_{H_0}$$

$$\underbrace{+ (g_{pair} - g_{pp}^{T=0}) \sum_{M_S=-1,0,1} A^\dagger_{1,0}(M_S,0)A_{1,0}(M_S,0) + (g_{pair} - g_{pp}^{T=1}) A^\dagger_{0,1}(0,0)A_{0,1}(0,0)\,.}_{H_I}$$

H_I violates SU(4) symmetry

g_{pair}- strength of isovector like nucleon pairing (L=0, S=0, T=1, M_T=±1)
$g_{pp}^{T=1}$- strength of isovector spin-0 pairing (L=0, S=0, T=1, M_T=0
$g_{pp}^{T=0}$- strength of isoscalar spin-1 pairing (L=0, S=1, T=0)
g_{ph}- strength of particle-hole force

M_F and M_{GT} do not depend on the mean-field part of H and are governed by a weak violation of the **SU(4)** symmetry by the particle-particle interaction of H

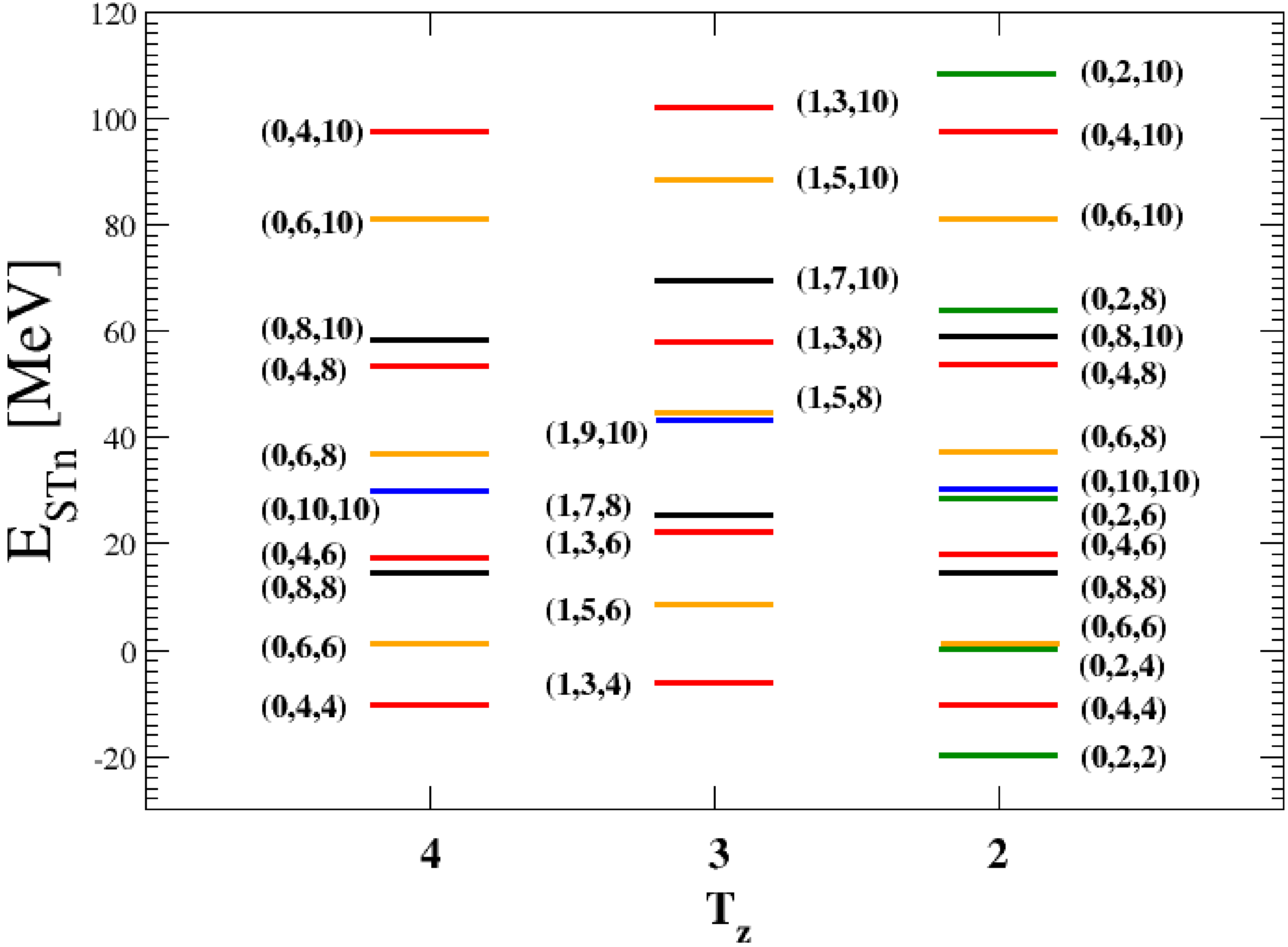
$$M_F^{2\nu} = -\frac{48\sqrt{\frac{33}{5}}\,(g_{pair} - g_{pp}^{T=1})}{(5g_{pair} + 3g_{ph})(10g_{pair} + 6g_{ph})}$$

$$M_{GT}^{2\nu} = \frac{144\sqrt{\frac{33}{5}}}{5g_{pair} + 9g_{ph}}\left\{\frac{(g_{pair} - g_{pp}^{T=0})}{(10g_{pair} + 20g_{ph})} + \frac{2g_{ph}(g_{pair} - g_{pp}^{T=1})}{(10g_{pair} + 20g_{ph})(10g_{pair} + 6g_{ph})}\right\}$$

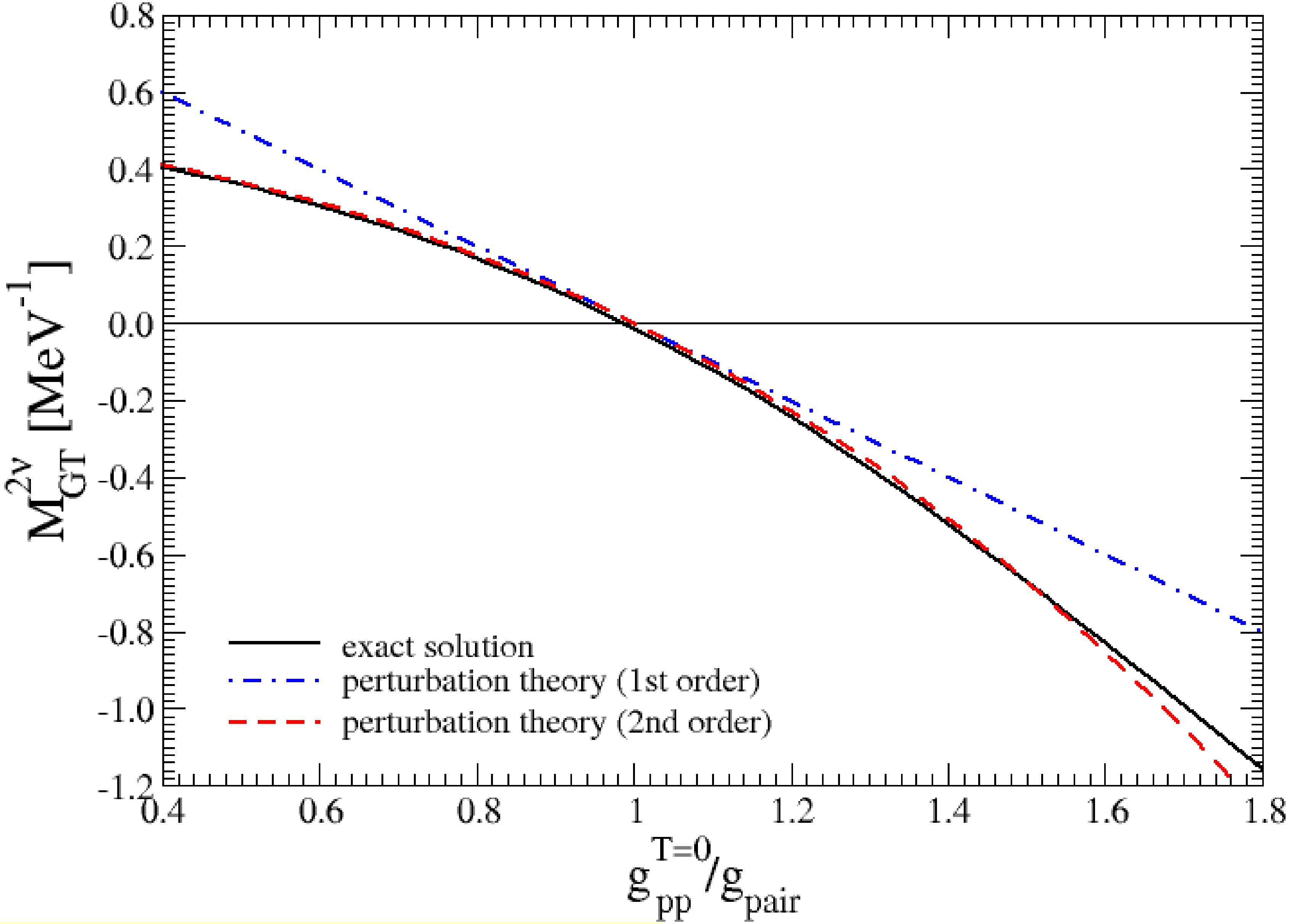
D. Štefánik, F.Š., A. Faessler, PRC 91, 064311 (2015)

Energies of excited states for the case of conserved SU(4) symmetry

M_F=0, M_{GT}=0 (see SU(4) multiplets)

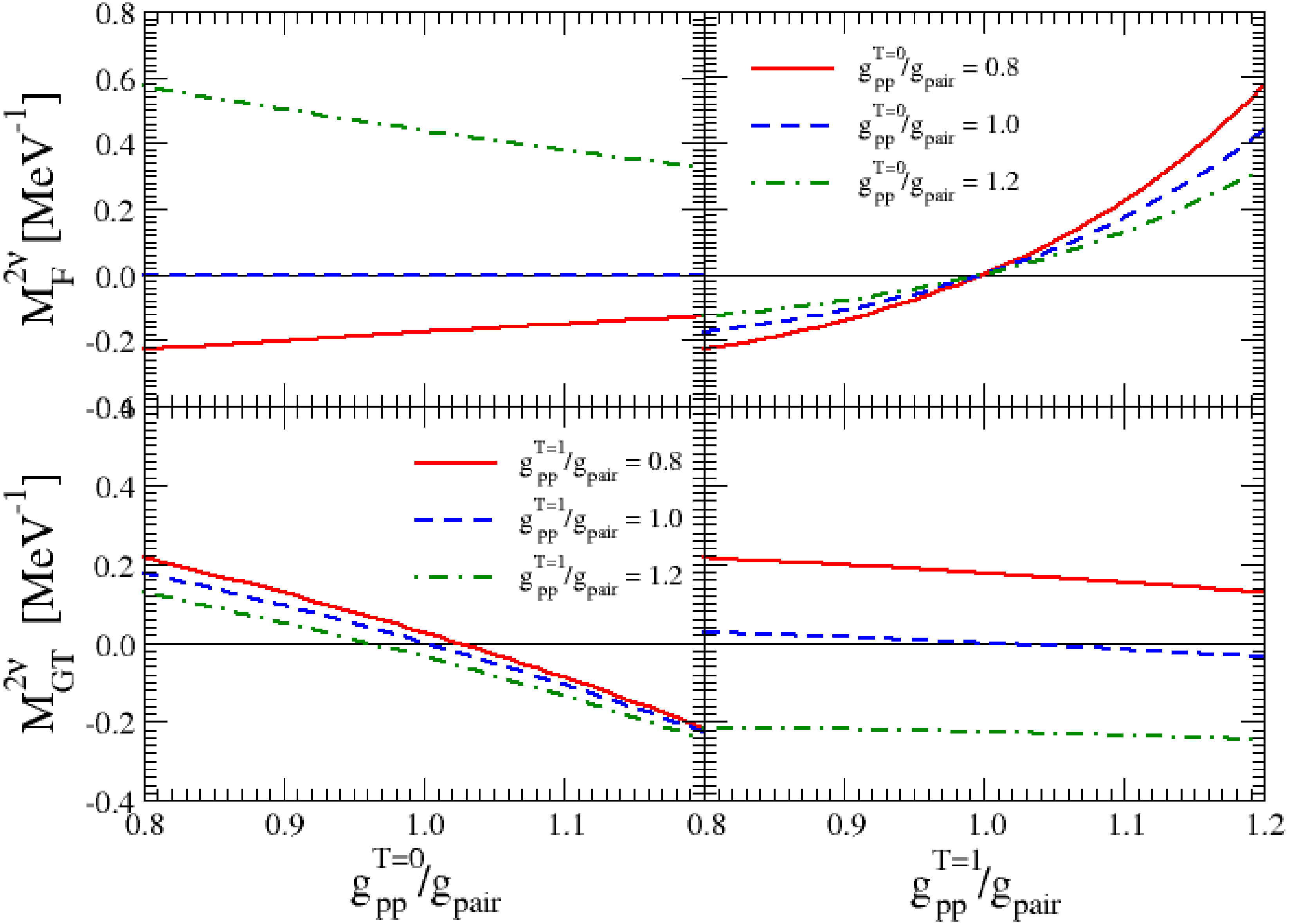


M_{GT} up to the second order of perturbation theory due to violation of the SU(4) symmetry by the particle-particle interaction of H

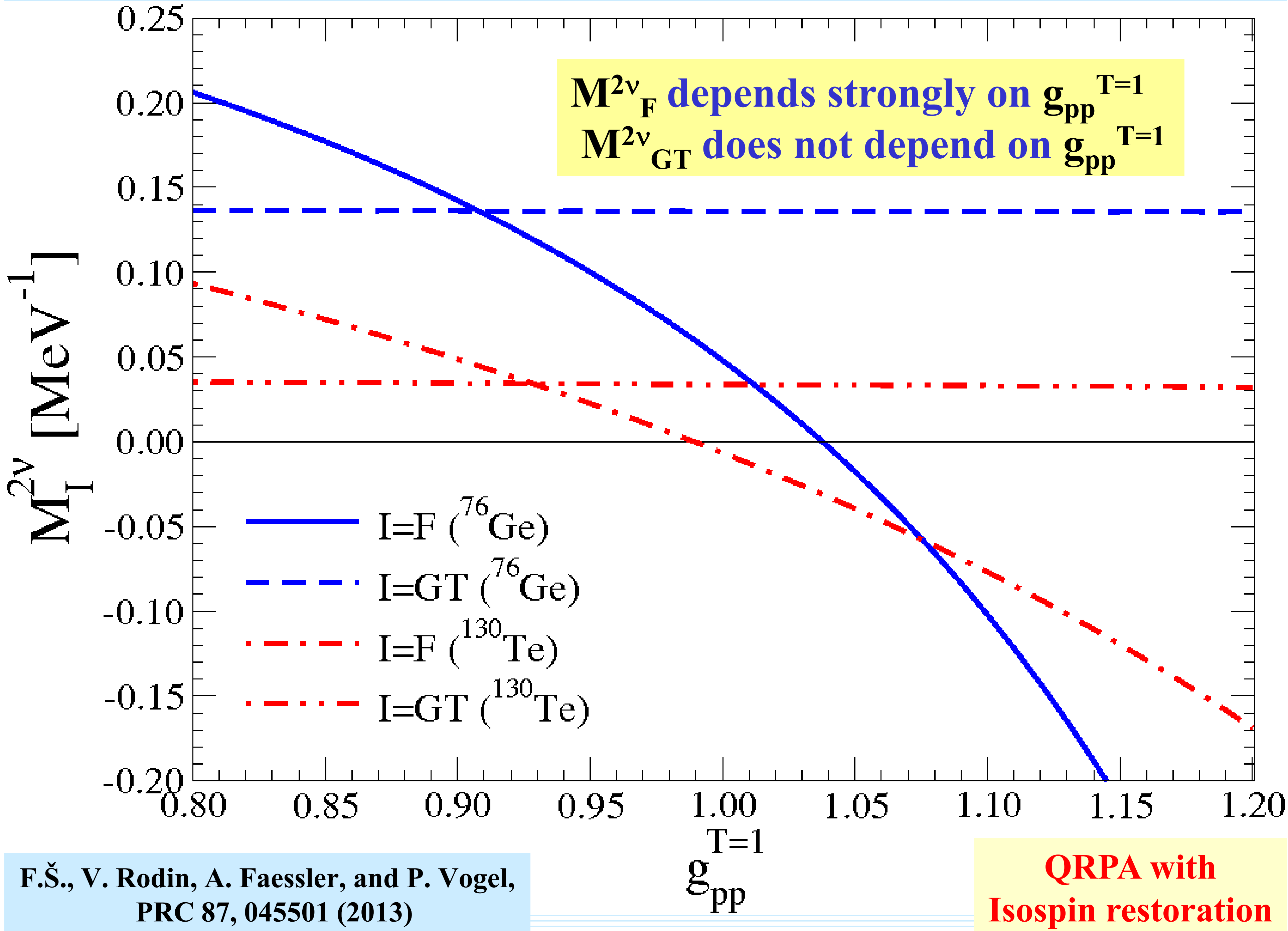


D. Štefánik, F.Š., A. Faessler, PRC 91, 064311 (2015)

Results confirm dependence of M_F and M_{GT} on $g_{pp}^{T=0}$ and $g_{pp}^{T=1}$ by the QRPA



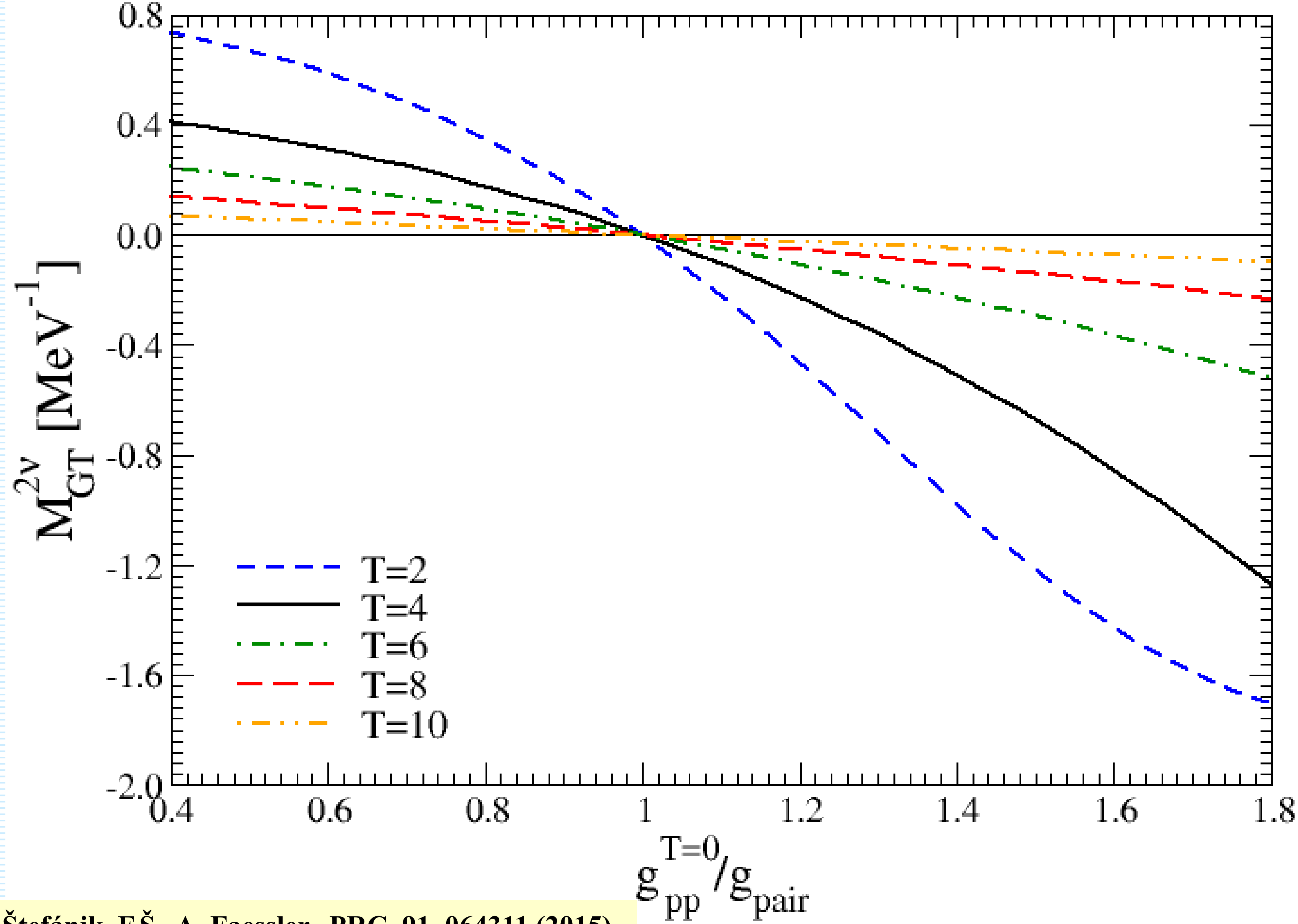
D. Štefánik, F.Š., A. Faessler, PRC 91, 064311 (2015)



F.Š., V. Rodin, A. Faessler, and P. Vogel, PRC 87, 045501 (2013)

QRPA with Isospin restoration

By assuming a fixed violation of the SU(4) symmetry by particle-particle int.
M_{GT} decreases by increase of isospin of the ground state



D. Štefánik, F.Š., A. Faessler, PRC 91, 064311 (2015)

Energy weighted sum rules of Δ=2 nuclei

$$S_F^{ew}(i,f) \equiv \sum_n \left(E_n - \frac{E_i + E_f}{2}\right) \langle f|\, T^- \,|n\rangle \langle n|\, T^- \,|i\rangle$$

$$= \frac{1}{2} \langle f|\, [T^-,[H,T^-]]\, |i\rangle ,$$

$$S_{GT}^{ew}(i,f) \equiv \sum_n \left(E_n - \frac{E_i + E_f}{2}\right) \langle f|\, \vec{\mathcal{O}}_{GT} \,|n\rangle \langle n|\, \vec{\mathcal{O}}_{GT} \,|i\rangle$$

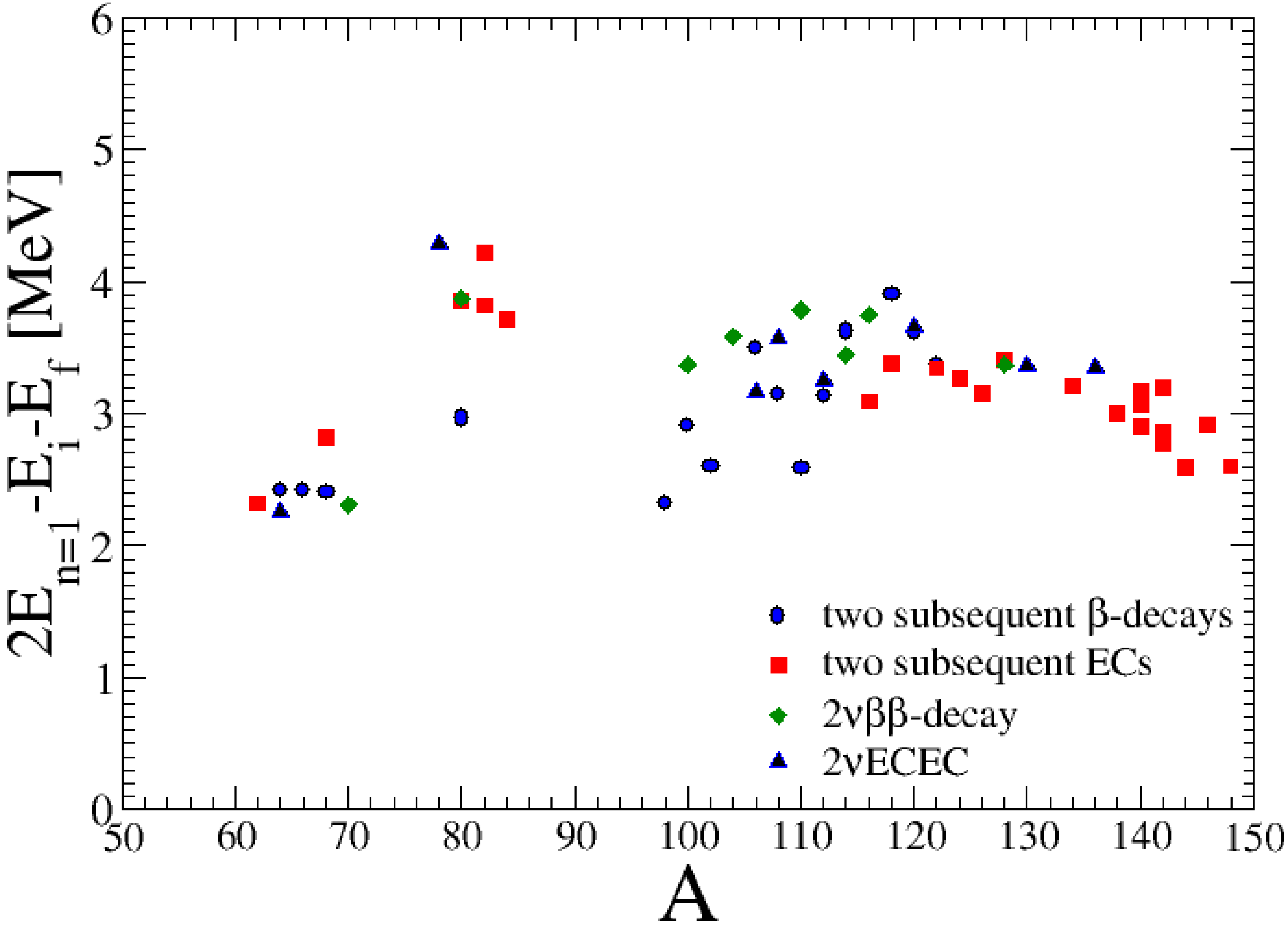
$$= \sum_M (-1)^M \frac{1}{2} \langle f|\, [(\mathcal{O}_{GT})_{-M},[H,(\mathcal{O}_{GT})_M]]\, |i\rangle ,$$

$$\vec{\mathcal{O}}_{GT} = \sum_{k=1}^{A} \tau_k^- \vec{\sigma}_k$$

$$E'_n - \frac{E'_i + E'_f}{2} = a g_{\text{pair}} + b g_{ph} + c\left(g_{\text{pair}} - g_{pp}^{T=1}\right) + d\left(g_{\text{pair}} - g_{pp}^{T=0}\right)$$

$T = M_T$	Transition	Coefficients			
		a	b	c	d
2	GT	3	5	−59/15	44/5
	Fermi	3	3	50/3	−59/5
4	GT	5	9	−64/35	39/5
4	Fermi	5	3	401/35	−192/35
6	GT	7	13	−71/63	340/63
	Fermi	7	3	482/63	−71/21
8	GT	9	17	−80/99	103/33
	Fermi	9	3	469/99	−80/33
10	GT	11	21	−7/11	12/11
	Fermi	11	3	26/11	−21/11

What is the meaning of quantity $(2E_{n=1}-E_i-E_f)$?

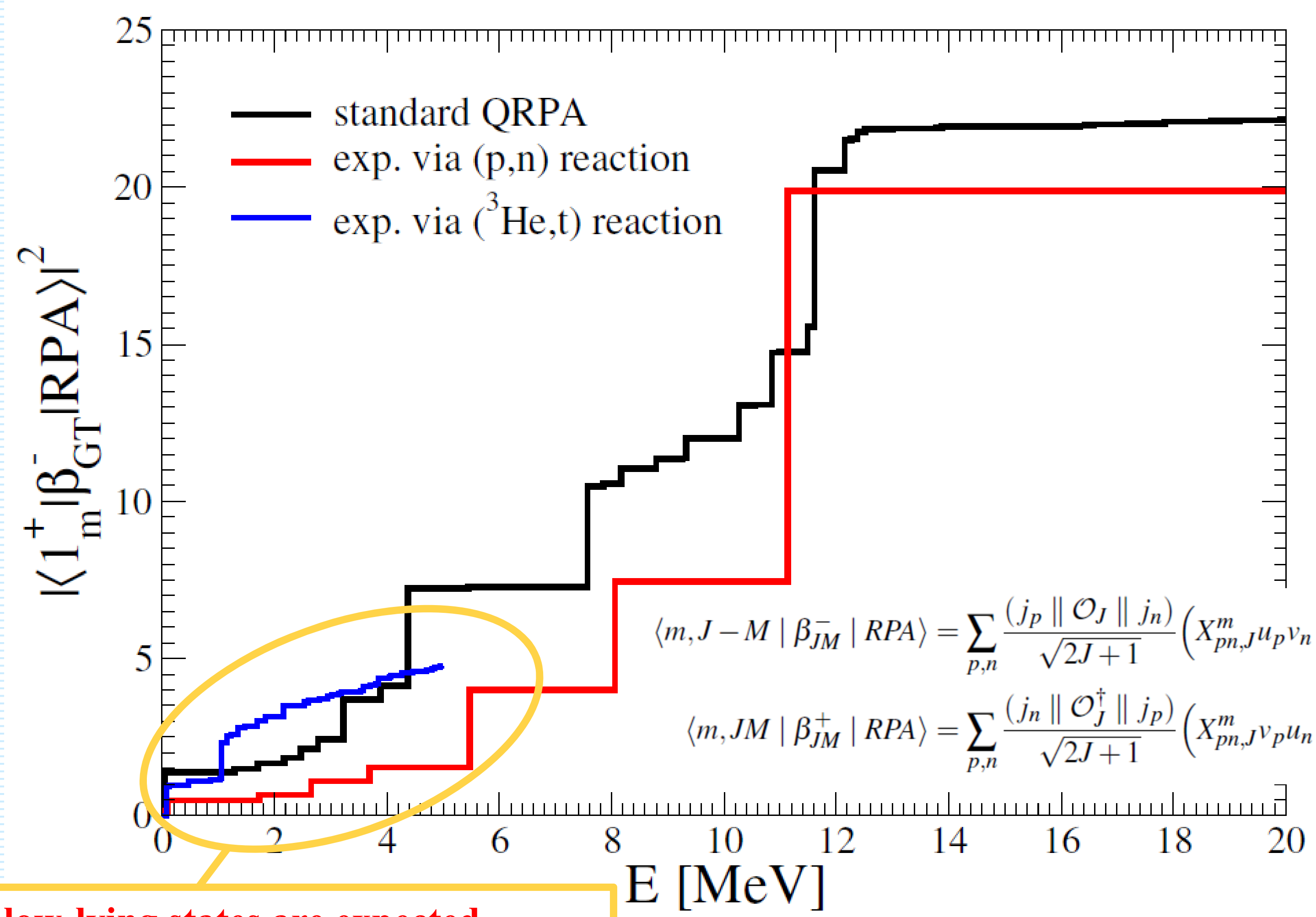


QRPA for description of states of multiphonon origin

A. Smetana, F.Š., M. Macko, AIP Conf. Proc. 1686, 020022 (2015) and to be submitted

β^- transitions in the standard QRPA

Calculate what can be confronted with experiment.



$$\langle m, J-M \mid \beta^-_{JM} \mid RPA \rangle = \sum_{p,n} \frac{(j_p \parallel \mathcal{O}_J \parallel j_n)}{\sqrt{2J+1}} \left(X^m_{pn,J} u_p v_n - Y^m_{pn,J} v_p u_n \right),$$

$$\langle m, JM \mid \beta^+_{JM} \mid RPA \rangle = \sum_{p,n} \frac{(j_n \parallel \mathcal{O}^\dagger_J \parallel j_p)}{\sqrt{2J+1}} \left(X^m_{pn,J} v_p u_n - Y^m_{pn,J} u_p v_n \right).$$

- **low-lying states are expected to be important for 2νββ decay**
- **we need improvement in this region**

Limitations of the standard QRPA

We want to fix the following limitations of the standard QRPA:

1. **Due to the QBA Pauli principle is broken and the QRPA colapses for the higher values of coupling parameters, which might be of physical interest.**
2. **Excited states of multi-phonon structure are neglected. Only the linear terms in phonon operator are considered.**

Schematic model

Use exactly solvable model to test your ideas.

We demonstrate the insufficiency of the multi-phonon approx. by comparison with the exact solution.

pn—Lipkin model **has the structure of the realistic hamiltonian.**

single *J*-shell with semidegeneracy $\Omega = \frac{N_p + N_n}{2}$

$$H_F = \varepsilon C + \lambda_1 A^\dagger A + \lambda_2 (A^\dagger A^\dagger + AA)$$

$$C = \sum_m a^\dagger_{pm} a_{pm} + \sum_m a^\dagger_{pm} a_{pm},$$

$$A^\dagger = [a^\dagger_p a^\dagger_n]^{J=0},$$

$$\lambda_1 = 2[\chi'(u_p^2 v_n^2 + v_p^2 u_n^2) - \kappa'(u_p^2 u_n^2 + v_p^2 v_n^2)]$$

$$\lambda_2 = 2(\chi' + \kappa') u_p v_p u_n v_n$$

κ' **parametrizes particle-particle and** χ' **parametrizes particle-hole interactions**

Schematic model – exact solution

The even and odd states do not mix!

Results are obtained from diagonalization of Hamiltonian.

basis of states:

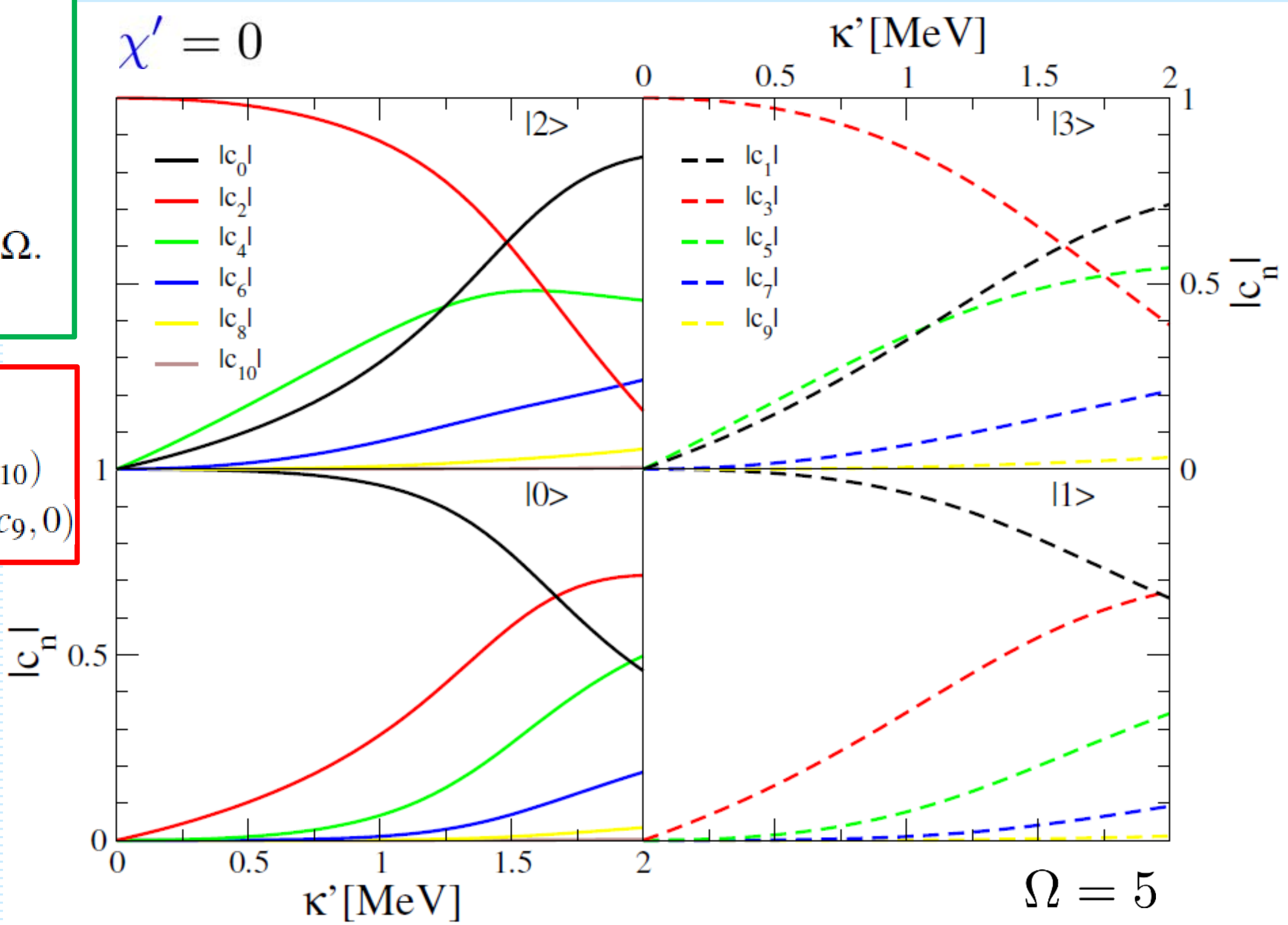
$$|\,n\rangle = \sum_{n'=0}^{2\Omega} c_{nn'}\,|\,n'_F\rangle$$

$$|n_F\rangle = \frac{(A^\dagger)^n}{\sqrt{m_n}}|0\rangle, \quad 0 \le n \le 2\Omega.$$

odd and even eigenstates:

$$|\,2i\rangle = (c_0, 0, c_2, 0, c_4, 0, c_6, 0, c_8, 0, c_{10})$$

$$|\,2i+1\rangle = (0, c_1, 0, c_3, 0, c_5, 0, c_7, 0, c_9, 0)$$

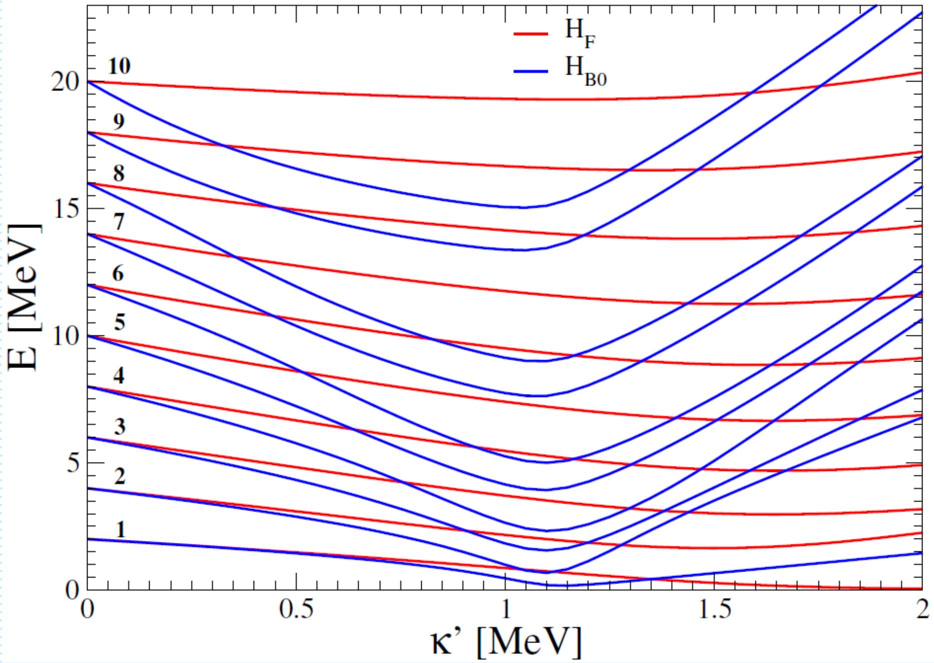


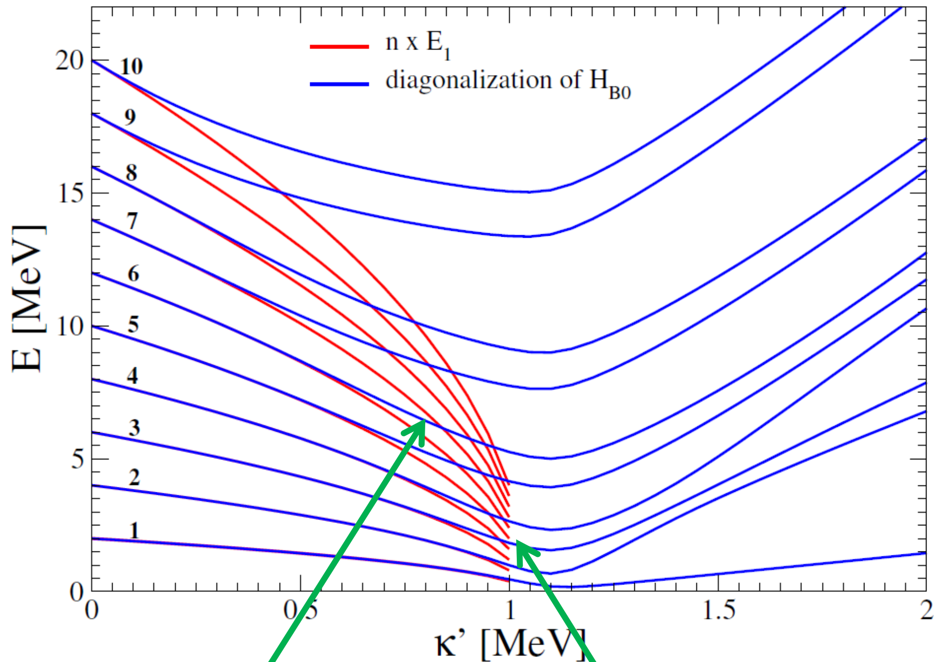
Schematic model – energy spectrum

$$H_F = \varepsilon C + \lambda_1 A^\dagger A + \lambda_2 (A^\dagger A^\dagger + AA) \xrightarrow{\text{QBA}} H_B = (2\varepsilon + \lambda_1) B^\dagger B + \lambda_2 (B^\dagger B^\dagger + BB)$$

exact Fermion model **vs** **exact QBA model**
vs **multi-phonon approximation**

$$|n\rangle \approx \frac{\left(Q^\dagger\right)^n}{n!} |RPA\rangle$$



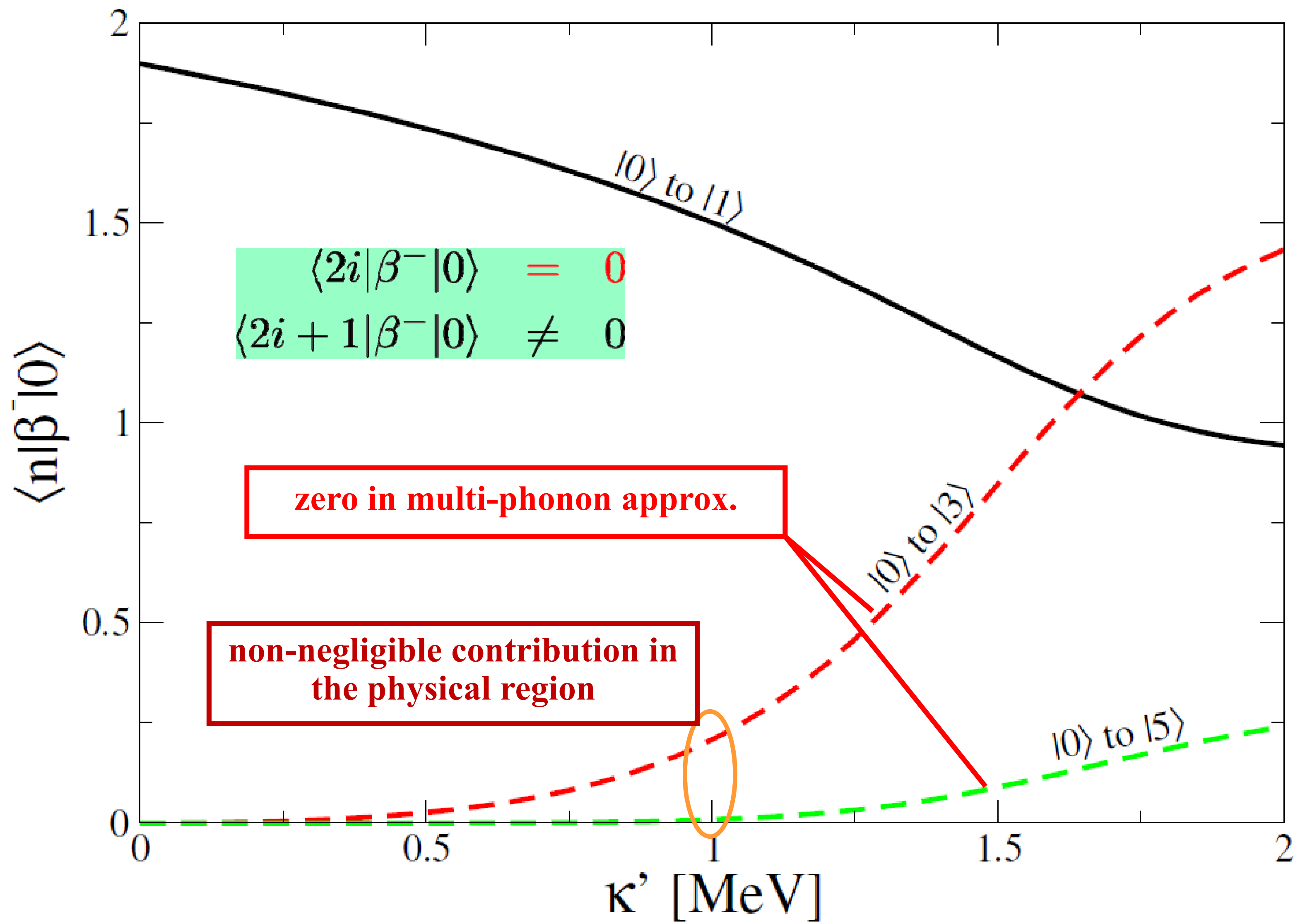


- **multi-phonon approach gives poor agreement for higher excited states**
- **standard QRPA is built for the first excited state only**

colapse of QRPA

Schematic model – β⁻ transitions

The multi-phonon approximation cannot reproduce the exact solution!

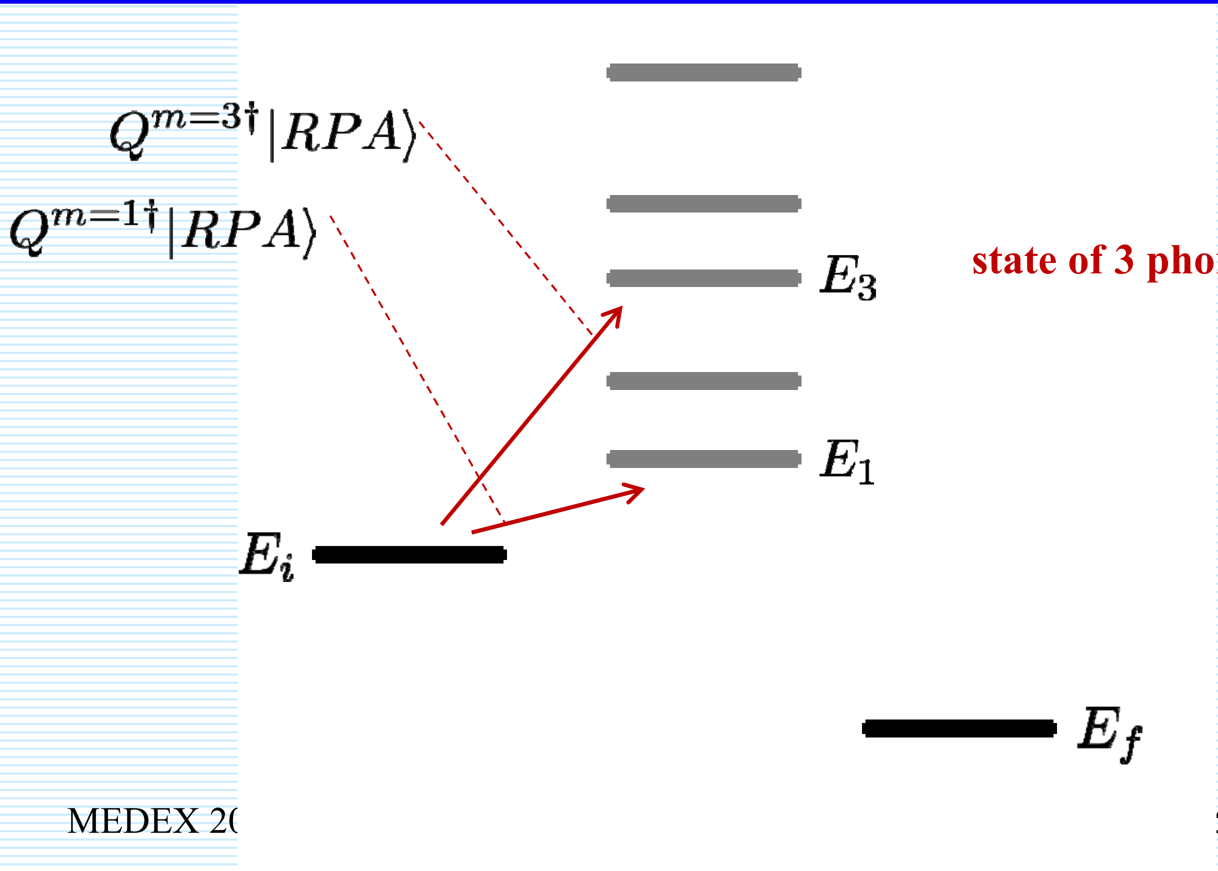


Idea of nonlinear phonon operator

Desired first goal: **the first and higher excited states described by single QRPA equation**

We introduce non-linear phonon operator: **QBA**

$$Q^\dagger_m = X^m_1 B^\dagger - Y^m_1 B + X^m_3 B^\dagger B^\dagger B^\dagger - Y^m_3 BBB$$



state of 3 phonon (lin. op.) origin

QRPA with non-linear phonon operator

The QRPA equation:

$$\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{13} & \mathcal{B}_{11} & \mathcal{B}_{13} \\ \mathcal{A}_{31} & \mathcal{A}_{33} & \mathcal{B}_{31} & \mathcal{B}_{33} \\ \mathcal{B}_{11} & \mathcal{B}_{31} & \mathcal{A}_{11} & \mathcal{A}_{13} \\ \mathcal{B}_{13} & \mathcal{B}_{33} & \mathcal{A}_{31} & \mathcal{A}_{33} \end{pmatrix} \begin{pmatrix} X_1^m \\ X_3^m \\ Y_1^m \\ Y_3^m \end{pmatrix} = E^m \begin{pmatrix} \mathcal{U}_{11} & \mathcal{U}_{13} & 0 & 0 \\ \mathcal{U}_{31} & \mathcal{U}_{33} & 0 & 0 \\ 0 & 0 & -\mathcal{U}_{11} & -\mathcal{U}_{13} \\ 0 & 0 & -\mathcal{U}_{31} & -\mathcal{U}_{33} \end{pmatrix} \begin{pmatrix} X_1^m \\ X_3^m \\ Y_1^m \\ Y_3^m \end{pmatrix}$$

Even in QBA approximation the norm matrix has not the standard form.

The RPA vacuum gets very complicated!!!

$$|RPA\rangle = \mathcal{N} \sum a_{2n} (B^\dagger B^\dagger)^n |0\rangle$$

the first 4 terms in the expansion of the RPA vacuum:

$$\mathcal{A},\ \mathcal{B},\ \mathcal{U}\ \propto\ a_2,\ a_4,\ a_6$$

$$a_0 = 1, \qquad a_2 = \frac{1}{2} \frac{(X_1^m X_1^m Y_1^m - 6 X_1^m X_3^m Y_3^m + 5 X_3^m Y_1^m Y_1^m)}{(X_1^m X_1^m X_1^m + 8 X_1^m X_3^m Y_1^m - 60 X_3^m X_3^m Y_3^m)},$$

$$a_4 = \frac{1}{8} \frac{(2 X_1^m X_1^m Y_3^m + X_1^m Y_1^m Y_1^m - 20 X_3^m Y_1^m Y_3^m)}{(X_1^m X_1^m X_1^m + 8 X_1^m X_3^m Y_1^m - 60 X_3^m X_3^m Y_3^m)}, \quad a_6 = \frac{1}{48} \frac{(6 X_1^m Y_1^m Y_3^m - 24 X_3^m Y_3^m Y_3^m + Y_1^m Y_1^m Y_1^m)}{(X_1^m X_1^m X_1^m + 8 X_1^m X_3^m Y_1^m - 60 X_3^m X_3^m Y_3^m)}$$

Need for further approximations and for constructing a closed iterative procedure.

QRPA with non-linear phonon operator

In every step of iteration we do:

1. **convert the norm matrix to its standar form…**

$$\begin{pmatrix} U & 0 \\ 0 & -U \end{pmatrix} = \mathcal{O}\lambda^{1/2} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \lambda^{1/2}\mathcal{O}^{-1}$$

…obtaining the parameters: its rotational angle & eigenvalues

2. **which are used to „rotate“ the system into the standard QRPA form**

$$\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{A} \end{pmatrix} \begin{pmatrix} \bar{X}^m \\ \bar{Y}^m \end{pmatrix} = E_m \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \bar{X}^m \\ \bar{Y}^m \end{pmatrix}$$

$$\lambda^{-1/2}\mathcal{O}^{-1} \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mathcal{O}\lambda^{-1/2}$$

$$\lambda^{1/2}\mathcal{O}^{-1} \begin{pmatrix} X^m \\ Y^m \end{pmatrix}$$

QRPA with non-linear phonon operator

…and we „linearize" the procedure

introducing F-operators to write the phonon operator in its „linear" form:

$$Q^\dagger_m = \overline{X}^m_1 F^\dagger_1 + \overline{X}^m_3 F^\dagger_3 - \overline{Y}^m_1 F_1 - \overline{Y}^m_3 F_3$$

where:

$$F^\dagger_1 = \frac{1}{\sqrt{\lambda_1}}\left(B^\dagger \cos\theta + B^\dagger B^\dagger B^\dagger \sin\theta\right), \qquad F_1 = \frac{1}{\sqrt{\lambda_1}}\left(B\cos\theta + BBB\sin\theta\right)$$

$$F^\dagger_3 = \frac{1}{\sqrt{\lambda_3}}\left(-B^\dagger \sin\theta + B^\dagger B^\dagger B^\dagger \cos\theta\right), \qquad F_3 = \frac{1}{\sqrt{\lambda_3}}\left(-B\sin\theta + BBB\cos\theta\right)$$

assuming QBA for F-operators

$$[F_1, F^\dagger_1] \simeq \langle RPA|[F_1, F^\dagger_1]|RPA\rangle = 1, \qquad [F_3, F^\dagger_3] \simeq \langle RPA|[F_3, F^\dagger_3]|RPA\rangle = 1,$$

$$[F_1, F^\dagger_3] \simeq \langle RPA|[F_1, F^\dagger_3]|RPA\rangle = 0, \qquad [F_3, F^\dagger_1] \simeq \langle RPA|[F_3, F^\dagger_1]|RPA\rangle = 0,$$

$$[F_1, F_3] \simeq \langle RPA|[F_1, F_3]|RPA\rangle = 0, \qquad [F^\dagger_1, F^\dagger_3] \simeq \langle RPA|[F^\dagger_1, F^\dagger_3]|RPA\rangle = 0.$$

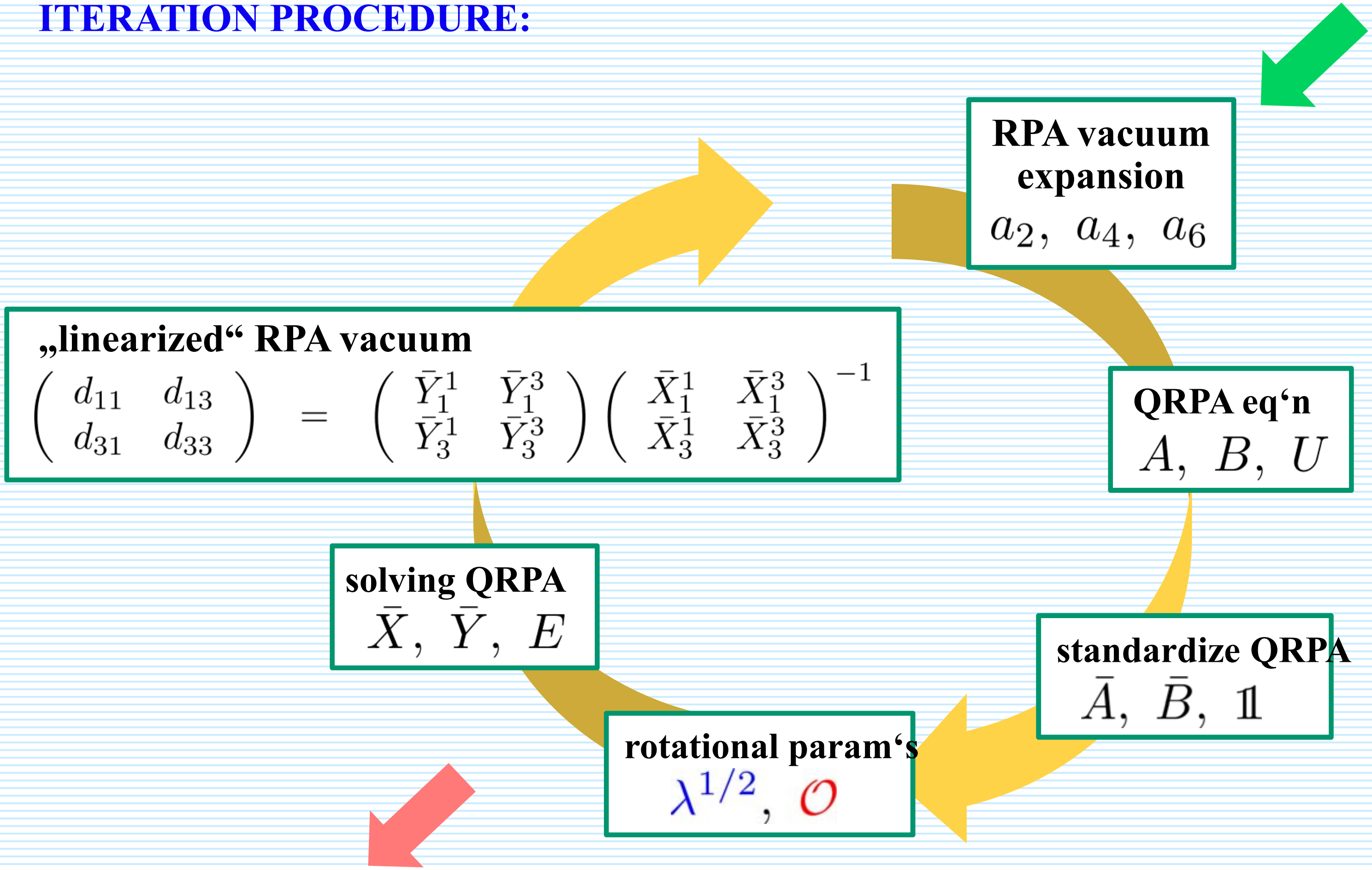
it allows us to construct standard-like RPA vacuum:

$$\begin{aligned} |RPA\rangle &= \mathcal{N}\, e^{\frac{1}{2}\sum_{ij} d_{ij} F^\dagger_i F^\dagger_j} |BCS\rangle \\ &\simeq \mathcal{N}\, \left(1 + a_2 B^\dagger B^\dagger + a_4 (B^\dagger B^\dagger)^2 + a_6 (B^\dagger B^\dagger)^3\right) \end{aligned}$$

QRPA with non-linear phonon operator

ITERATION PROCEDURE:

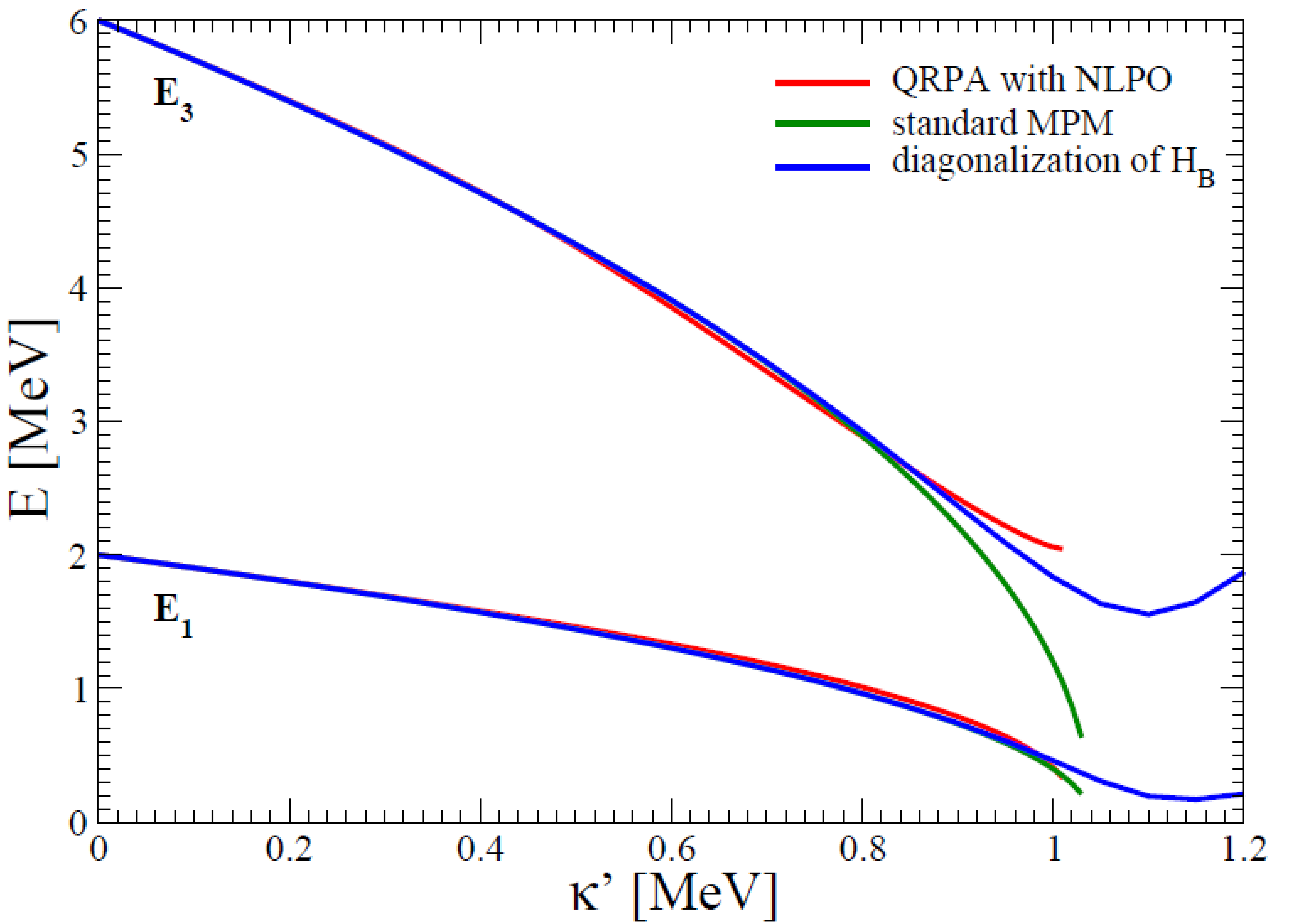
initial values



amplitudes and energies
if the iteration converges

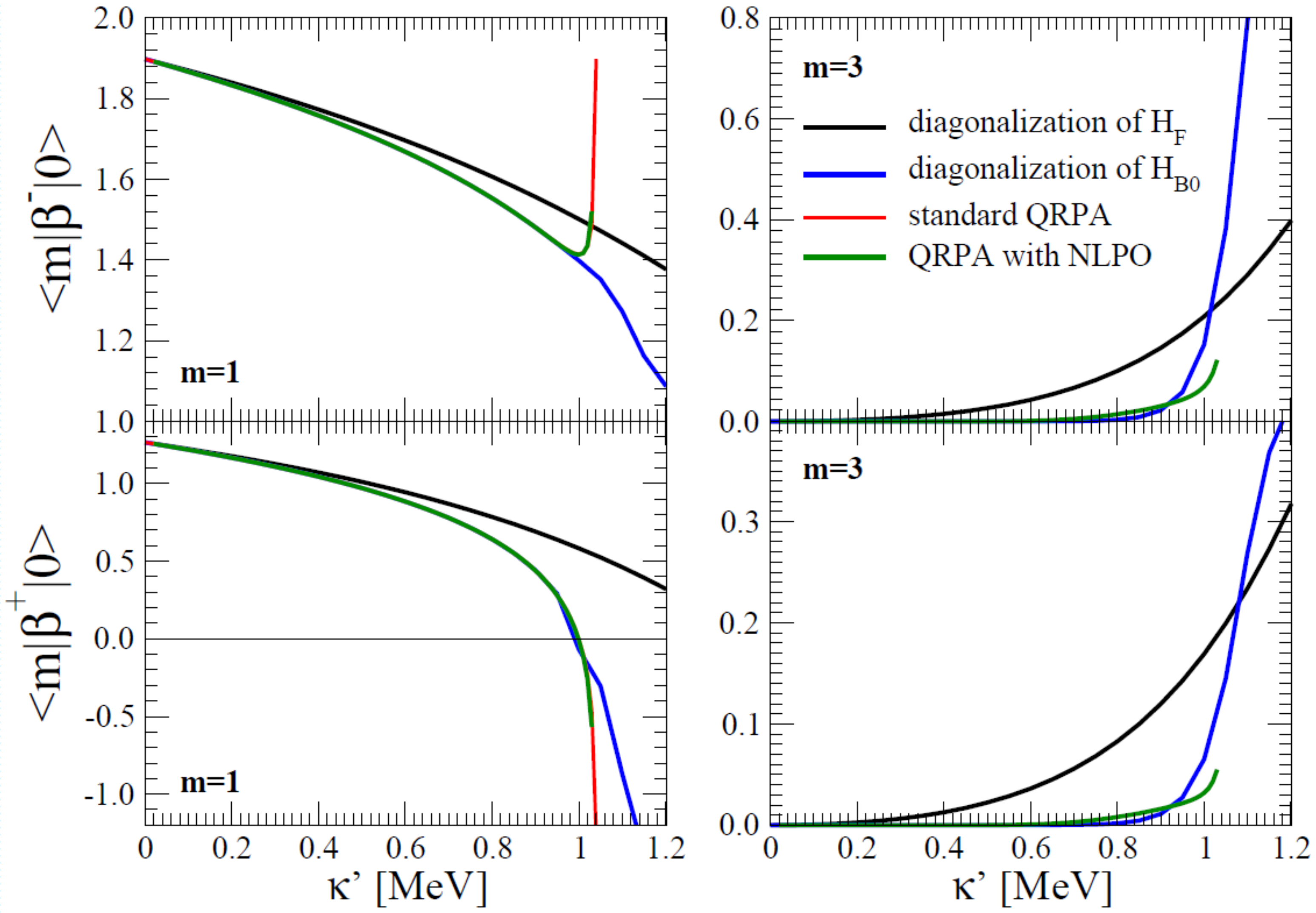
Results

- **Energies of the first and the third excited states**



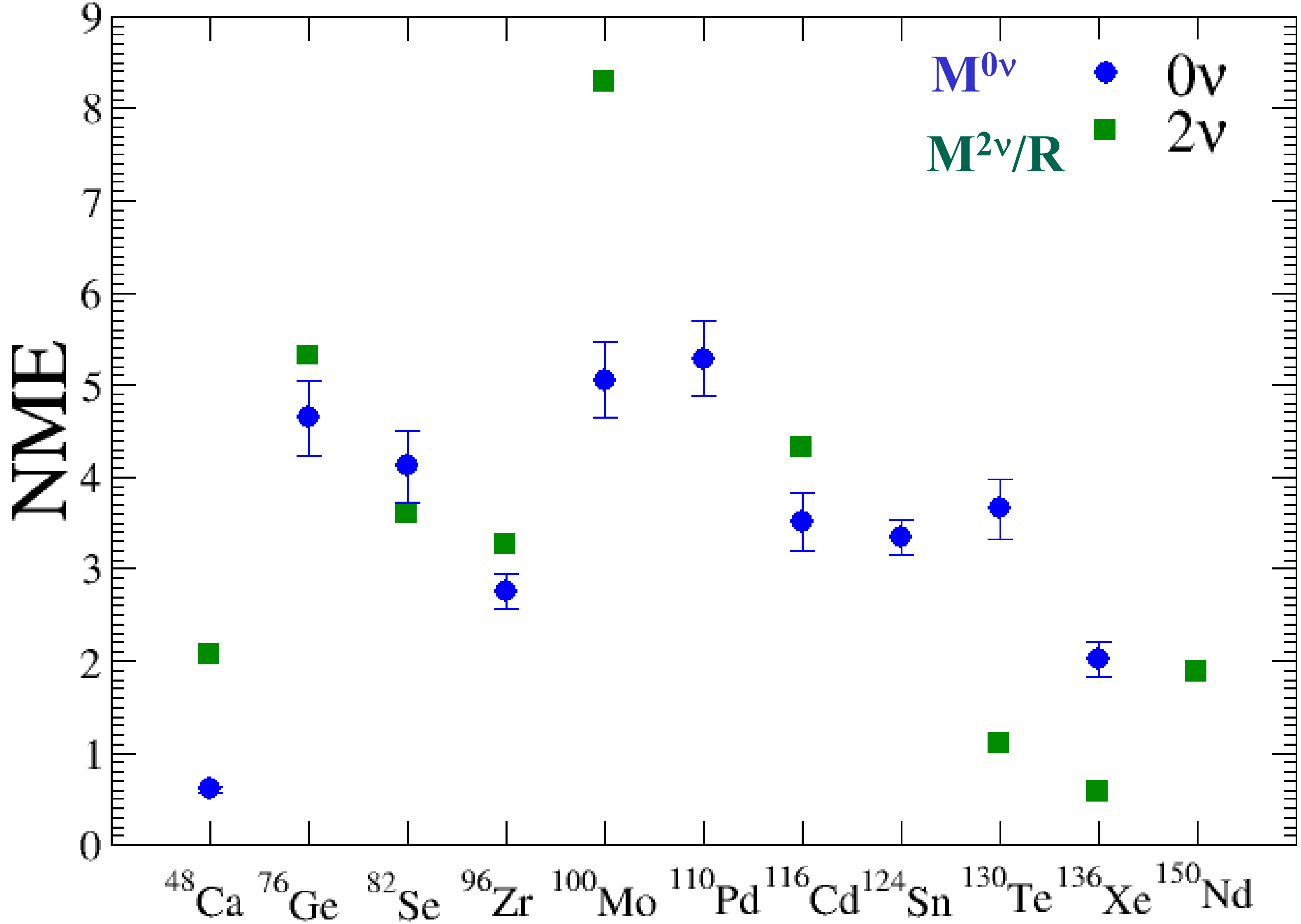
Results

- **beta transition operators**



Is there a scaling factor
between 0νββ- and 2νββ-decay NMEs?

F. Š., Nucl.Part.Phys.Proc. 265-266 (2015) 19-24



How many $0\nu\beta\beta$-decay NMEs have to be calculated?

$\mathbf{M_F, \; M_{GT}, \; M_T \ldots}$

The 0νββ-decay with emission of electrons in $s_{1/2}$ and $p_{1/2}$ wave state

D. Štefánik, R. Dvornický, F.Š., Nuclear Theory 33 (2014) 115

$$\psi(\mathbf{r},p,s) \simeq \psi_{s_{1/2}}(\mathbf{r},p,s) + \psi_{p_{1/2}}(\mathbf{r},p,s) =$$

$$\begin{pmatrix} g_{-1}(\varepsilon,r)\chi_s \\ f_{+1}(\varepsilon,r)\,(\vec{\sigma}\cdot\hat{\mathbf{p}})\,\chi_s \end{pmatrix} + \begin{pmatrix} ig_{+1}(\varepsilon,r)\,(\vec{\sigma}\cdot\hat{\mathbf{r}})\,(\vec{\sigma}\cdot\hat{\mathbf{p}})\,\chi_s \\ -if_{-1}(\varepsilon,r)\,(\vec{\sigma}\cdot\hat{\mathbf{r}})\,\chi_s \end{pmatrix}$$

Exact relativ. electron w.f.

$$J^\rho(\mathbf{x}) = \sum_n \tau_n^+ \delta(\mathbf{x}-\mathbf{r}_n)\left[(g_V - g_A C_n)\, g^{\rho 0} + g^{\rho k}\right.$$

$$\left.\times\left(g_A\sigma_n^k - g_V D_n^k - g_P\left(p_n^k - p_n^{'k}\right)\frac{\vec{\sigma}_n\cdot\left(\mathbf{p}_n - \mathbf{p}_n^{'}\right)}{2m_N}\right)\right]$$

Higher order terms of nucleon current with nucleon recoil

$$C_n = \frac{\vec{\sigma}_n\cdot\left(\mathbf{p}_n + \mathbf{p}_n^{'}\right)}{2m_N} - \frac{g_P}{g_A}\left(E_n - E_n^{'}\right)\frac{\vec{\sigma}_n\cdot\left(\mathbf{p}_n - \mathbf{p}_n^{'}\right)}{2m_N}$$

$$\mathbf{D}_n = \frac{\left(\mathbf{p}_n + \mathbf{p}_n^{'}\right)}{2m_N} - i\left(1 + \frac{g_M}{g_V}\right)\frac{\vec{\sigma}_n\times\left(\mathbf{p}_n - \mathbf{p}_n^{'}\right)}{2m_N}.$$

0νββ-decay rate with $p_{1/2}$ electrons (2 additional NMEs and 5 phase-space factors)

$$\left[T_{1/2}^{0\nu\beta\beta}\right]^{-1} = \frac{|m_{\beta\beta}|^2}{m_e^2} g_A^4 \left(2Re\{M_s M_r^*\} G_{sr} + 2Re\{M_s M_p^*\} G_{sp} + 2Re\{M_r M_p^*\} G_{rp} + G_{ss}|M_s|^2 + G_{rr}|M_r|^2 + G_{pp}|M_p|^2\right),$$

$$M_s = -\frac{M_F}{g_A^2} + M_{GT} + M_T \qquad M_{F,GT,T} = \sum_{r,s} \langle 0| h_{F,GT,T}(r_-) \mathcal{O}_{\mathcal{F},\mathcal{GT},\mathcal{T}} |0\rangle$$

$$M_p = -\frac{M'_F}{g_A^2} + M'_{GT} + M'_T + M_V + M_A + M'_A$$

$$M_V = i \sum_{r.s} \langle 0| \frac{h_{AV}(r_-) + h_{VP}(r_-)}{2R^2} \tau_r^+ \tau_s^+ (\mathbf{r}_- \times \mathbf{r}_+) \cdot \vec{\sigma}_r |0\rangle$$

$$M'_{F,GT,T} = \sum_{r.s} \langle 0| h_{F,GT,T}(r_-) \mathcal{O}_{\mathcal{F},\mathcal{GT},\mathcal{T}} \left(\frac{|\mathbf{r}_-|^2 - |\mathbf{r}_+|^2}{4R^2}\right)$$

$$M_A = \sum_{r,s} \langle 0| \frac{h_{AP}(r_-) + h_{AA}(r_-) + h_{MM}(r_-)}{2R^2} \times \tau_r^+ \tau_s^+ (\vec{\sigma}_r \cdot \mathbf{r}_-)(\vec{\sigma}_s \cdot \mathbf{r}_+) |0\rangle$$

$$M'_A = \sum_{r,s} \langle 0| h_A(r_-) \frac{(\mathcal{O}_{\mathcal{T}} + \mathcal{O}_{\mathcal{GT}})\, \mathbf{r}_- \cdot \mathbf{r}_+}{2R^2} |0\rangle$$

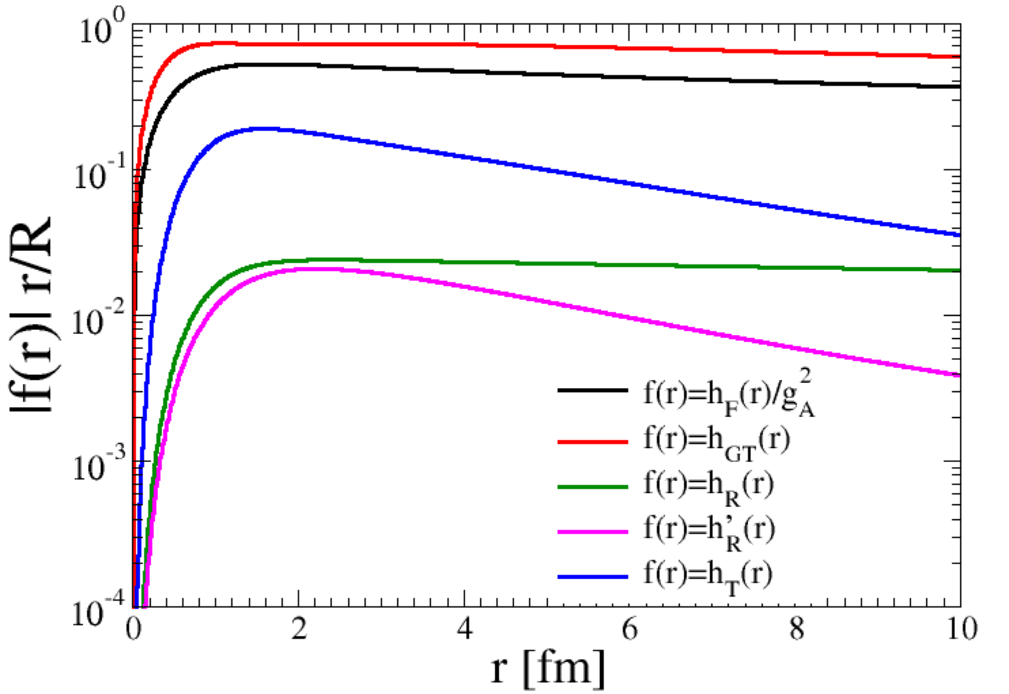
$$M_r = \sum_{r.s} \langle 0| \left(h_R(r_-) + h'_R(r_-)\right) \mathcal{O}_{\mathcal{T}} - 2h_R(r_-) \mathcal{O}_{\mathcal{GT}} |0\rangle$$

	^{48}Ca	^{76}Ge	^{82}Se	^{96}Zr	^{100}Mo	^{110}Pd	^{116}Cd	^{124}Sn	^{130}Te	^{136}Xe	^{150}Nd
$Q_{\beta\beta}$ [MeV]	4.27226	2.03904	2.99512	3.35037	3.03440	2.01785	2.8135	2.28697	2.52697	2.45783	3.37138
G_{ss} $[10^{-18} yr^{-1}]$	24 834.	2 368.1	10 176.	20 621.	15 953.	4 828.5	16 734.	9 063.5	14 255.	14 619.	63 163.
G_{sr} $[10^{-18} yr^{-1}]$	−4 138.3	−529.26	−2 499.4	−5 929.3	−4 738.2	−1 504.8	−5 569.5	−3 082.8	−5 071.1	−5 385.7	−26 409.
G_{rr} $[10^{-18} yr^{-1}]$	690.26	118.37	614.25	1 705.7	1 407.9	469.16	1 854.5	1 049.0	1 804.7	1 984.9	11 045.
G_{sp} $[10^{-18} yr^{-1}]$	−171.01	−29.513	−152.98	−424.86	−350.88	−117.07	−462.44	−261.74	−450.22	−495.23	−2 754.1
G_{rp} $[10^{-18} yr^{-1}]$	28.553	6.6047	37.619	122.29	104.31	36.518	154.05	89.101	160.29	182.59	1 152.3
G_{pp} $[10^{-18} yr^{-1}]$	1.1824	0.36878	2.3055	8.7718	7.7325	2.8437	12.802	7.5711	14.242	16.803	120.25

Calculated phase-space factors for 0νββ-decay with emission of $s_{1/2}$ and $p_{1/2}$ electrons

($m_{\beta\beta}$ mechanism)

	^{48}Ca	^{76}Ge	^{82}Se	^{96}Zr	^{100}Mo	^{110}Pd	^{116}Cd	^{124}Sn
				r = 1 fm				
$2h_R(r)/h_{GT}(r)$	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04
$\Delta[\%]$	0.31	0.55	0.93	1.81	2.03	2.44	4.22	3.24
				r = 2 fm				
$2h_R(r)/h_{GT}(r)$	0.71	0.07	0.06	0.06	0.06	0.06	0.06	0.06
$\Delta[\%]$	1.19	0.46	0.20	0.60	0.82	1.21	3.60	1.95



^{130}Te	^{136}Xe	^{150}Nd
0.04	0.04	0.03
3.70	4.10	5.74
0.06	0.05	0.05
2.37	2.75	4.27

Effect of $p_{1/2}$ wave is below 10%.

Understanding of the 2νbb-decay is the key for reliable calculation of 0νbb-decay NMEs